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Logic Of Complementarity As A Synthesis To Completeness And Incompleteness Theorems Of Godel

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Abstract
Mathematical logic is central to logic in the modern time. This mounts great emphasis on formal systems and arithmetical laws which are correlational to human reason but less regards to intuition. The aim of this paper is an attempt to draw a logical complementarity as a synthesis to the extremes of completeness and incompleteness theorems of Kurt Godel. Using the analytical method of research, this paper makes a critique of completeness and incompleteness theorems of Godel, hence exposing their limitations/inadequacies in the representation of reality in entirety. The paper studies completeness theorem as a theory which is both consistent and complete while incompleteness theorem demonstrates the fundamental inadequacies of formal systems. In the same vein, this papers studies formalism as the popular stand of logicians in the 19th century with particular concern on Hilbert. Also, the studies reviews the strength and weakness of both completeness and incompleteness theorems. By appraisal, this paper opines that a true logical position that represents the wholeness of reality and humans is a logical complementarity of both completeness and incompleteness theorems. Thus while completeness theorem shows the strength of formal systems correlational to reason, incompleteness theorem punctures the inadequacies of formal systems to address intuitive capacities of the human person beyond the provisions and provability of reason.

Keywords: Godel, completeness, incompleteness, formalism, intuition, etc.

Introduction
In the words of Neumanne, "Kurt Godel's achievement in modern logic is singular and monumental; a landmark which will remain visible far in space and time".[1] Kurt Godel was a renowned philosopher, a mathematician and an outstanding logician. Together with Aristotle[2] and Frege, the trio remain the most outstanding and significant logician in the epochs of philosophy. Godel significantly made mind blowing impact in the area of science and philosophy in the twentieth century[3]. Godel's specialty is in the area of mathematical logic. He made lots of discoveries in the area of foundations of mathematics which culminated to his first theorem, completeness theorem in 1929. This work was part of his doctorate dissertation at the university of Vienna[4]. In the same vein, two years afterwards in 1931, he presented his incompleteness theorem which addresses limitations of formal axiomatic systems[5]. Here he argues that formalism or axiomatic systems though technical is inadequate to satisfy or decide the truth condition or value of all statements about mathematics/natural numbers yet cannot prove its completeness and its consistency.[6]
This paper argues that complementrity remains a good synthesis of the two extremes of Godel's completeness and incompleteness theorems. There are some fundamental questions strengthening this proposal for complementarity of completeness and incompleteness theorems. The basic questions include; are there realities that are unreachable by the senses even with the sophistications of formalism? Do all realities obey the sophisticated principles of formalism or physical laws? The complexity of human senses regarding information management makes the senses a bundle of of possibilities

[1]. Halmos, P. (1973). "The Legend of Von Neumann", *The American Mathematical,* 80(4): 382-394.

[2]Aristotle, *Metaphysics,* Book XI, Part 5.
[3] Driessen, A. (1989). *Philosophical Consequences of the Godel Theorem,* (Netherland: University Press), 23.
[4] Smullyan, R. (1992). *Godel's Incompleteness Theorem.* (New York: Oxford University Press), 5.
[5]. Smullyan, R. (1992). *Godel's Incompleteness Theorem.* 5
[6].Smullyan, R. (1992). *Godel's Incompleteness Theorem.* 9

and as such can know by intuition beyond the capacities of physical laws and formal systems.

Completeness Theorem in Godel

According to Alan, a theory can be understood as a set of axioms. It can be further said to be mean basic statements which are used to deduce facts concerning specific domains[7]. A theory in the field of physics uses axioms to prove reoccurring patterns of physical realities in nature. A number theory is a branch of mathematics which focuses on studying the properties, relationship, and behaviors of integers. For logicians, a theorem is a mathematically proven proposition derived from axioms representing an absolute and logical truth. This study deals on theorem rather than theories.

As a renowned philosopher, Kurt Godel's landmark in the areas of mathematics gave a firm foundation to his completeness theorem in 1929 as part of his doctoral thesis at the University of Vienna. In the same vein, his Incompleteness theorem came two years afterwards[8] and the subsequent coding of natural numbers. Here Godel numbering is a technique from mathematical logic such that specific numbers or codes are assigned to symbols or formulas in order to ease self-reference and consistency of arithmetic system.

Godel's completeness theorem claims that in first order logic, any valid formula is provable. This implies that for a set of axioms, all logical consequences can be derived through formal deduction. In his completeness theorem, Godel maintains that ideally a theorem ought to be consistent and at the same time complete. A theorem is consistent if its axioms cannot be used to prove a contradiction.

In the same vein, a contradiction is a statement that presents opposite of its original state such as, A and -A. An instance of contradiction is; "he is a boy and he is not a boy" (B . -B). This is an undesirable condition in logic following the possibilities of the same theory proving as many falsehoods (a bunch of falsehood) as much as many true statements. a bunch of falsehoods in addition to a bunch of true statements. Also, the theorem is complete when its truth claims can be proven by itself. Thus this implies that for any instance, the theory or the opposite (its nagation) can also be proven by itself.

Fracture of Formalism in Incompleteness Theorem of Godel

In his popular work, "On Formally Undecidable Propositions of *Principia Mathematica* and Related Systems I" Godel demonstrated the fundamental inadequacy of the pure formal approach[9]. Here pure formal approach is associated to the works of Hilbert's strict formalism. Strict formalism is the belief that mathematics can be entirely reduced to a finite, consistent set of axioms and mechanical rules of deduction, where meaning is irrelevant and truth is synonymous with provability[10]. Thus Godel demonstrated that fundamental inadequacy of this approach that such a system cannot be both consistent and complete if it is complex enough to describe basic arithmetic[11]. For every axiomatic system Godel opines that;

1. There exists a true proposition which cannot be deduced from axiomatic systems. Godel refers to this as incompleteness theorem.
2. Consistency of the axioms cannot be derived from axiomatic systems.

This second position refers to Godel's consistency theorem.

What is obvious in the both theorems is that for Godel, there are certain mathematical truths that cannot be obtained by express application of the deduction. This means that certain mathematical truth cannot be obtained by means of formalism. Thus Godel desires not to reduce both mathematical truth to formal logic. There are obvious propositions whose truth validity and correctness are not dependable on mathematical axioms and can be clear enough to one who has a minimum understanding of mathematical education[12].

It is obvious that Godel's proposition is legitimate enough. There are two alternative perspectives; thus a proof for a proposition or no proof for a proposition. In the first alternative, a proof to legitimize Godel's claims on incompleteness theorem is when a proposition whose correctness followed axiomatic formulation of formal system is wrong. On the other hand where there is no proof to legitimize Godel's claims (second alternative) is when there no axiomatic proofs yet the proposition is true, hence the incompleteness theorem is correct. It is obvious that the second alternative cannot be said to have any form of contradiction because its truth validity is not assured by means of axiomatic formulations. A possible conclusion deducible from these claims is that the truth validity of propositions is undecidable and cannot completely be proven by means of formal systems of logic.

In furtherance of his attempt to legitimize the incompleteness theorem, Godel claims that axiomatic systems which can

[7]Alan, K. (2015). "Godel and Mathematical Logic". *Metaphysics Research Lab, Stanford University*. 7(21), 67-89.

[8]. Smullyan, R. (1992). *Godel's Incompleteness Theorem.* 5

[9]. Godel, K. (1931). *On Formally Undecidable Propositions of Principia Mathematica and Related System I*. (London: Dove Publications), 12.

[10]. Godel, K. (1931). *On Formally Undecidable Propositions of Principia Mathematica and Related System I*. 174.

[11]. Godel, K. (1931). *On Formally Undecidable Propositions of Principia Mathematica and Related System I*. 173

[12].Godel, K. (1931). *On Formally Undecidable Propositions of Principia Mathematica and Related System I*. 12

be said to be sufficiently powerful and consistent are not complete. By sufficiently powerful theories, Godel implies those theories that proof lots of facts concerning numbers and at the same time proof facts about themselves. An instance of axiomatic provability can be represented by arithmetical statements such as; 2+3=5.

In order to address the paradox of the ability of arithmetic theories proving facts about statements or numbers and then facts about themselves, Godel developed the coding method which is referred as, "Godel coding". As much as arithmetic theories are supposed to prove statements such as, 2+3=5, it is not expected that the theories prove things about themselves as they do to 2+3=5. This is because the latter statement is not simply about numbers but about the theory itself.[13]

Development of Godel code is an attempt towards interpreting arithmetic statement in order to define the theory itself. Godel code is constructed by the assignment of some codes to the various symbols of the theory. For instance to the arithmetical statement above, 2+3=5; there are five symbols in this formula such as, 2, +, 3, =, 5, hence each is assigned a Godel code which can be picked arbitrarily. Here we pick the following number; 11, 13, 15, 17, and 19. It is obvious that different symbols are ascribed different Godel codes. Thus we can represent it in the table as follows:

Symbol	2	+ (plus)	3	= (equals)	5
Godel number	11	13	15	17	19

For Godel, these numbers can be encoded using the first five prime numbers since they are five symbols in the arithmetical statement/slots, 2, +, 3, =, 5; hence the first five prime numbers to encode these symbols are 2, 3, 5, 7, 11. The coding entails raising each of these numbers to the power or superscript of their corresponding Godel numbers and arithmetic symbols in the table. In the first column of the table, the first prime number is 2 and to code this number, it is raised to the superscript or power of the corresponding first column Godel number 11, hence 2^{11}. Similarly, the second prime number 3 is coded by raising it to the superscript or power of the corresponding second column Godel number symbol +, hence 3^{13}. The same code formula is applicable to the third prime number 5 which is to be raised to the power of the Godel number 15 at the third symbol 3. This same method of coding applies to both the fourth and fifth prime numbers. After the coding, all are multiplied[14] to get a large sum;

$$2^{11} x 3^{13} \text{ x } 5^{15} x 7^{17} x 11^{19}$$

$$22 + 39 + 75 +1\ 19 + 209 = 464$$

Therefore, 464 is the Godel code for the formula 2 + 3 =5

The most outstanding input of Godel is a proof that no formal system is complete so much so that even mathematics has some axioms whose validity is improvable or undecidable by formal principles. Thus mathematics cannot be reduced to simple applications of formal steps of proof to accommodate matters of insight, intuition, meaning, etc.

Synthesis of Complimentarity Between Completeness and Incompleteness Theorems in Godel

In his attempt to defend completeness theorem, Godel claims that a theory ought to be both consistent and complete[15]. A theory is consistent if its axioms cannot provide grounds to establish contradictions. Also, a theory is complete when it provides sufficient proofs of its statements. However, in his incompleteness theorem, Godel claims that these theories whose axioms cannot provide grounds to establish contradictions and are said to be consistent are yet not complete.

Godel demonstrated the fundamental inadequacy of the pure formal approach[16], hence established the incompleteness theorem. Here he pledged his non acceptance of strict formalism in mathematics where truth becomes synonymous with provability[17]. Godel demonstrated the inadequacy of this approach claiming that a system cannot be both consistent and complete if it is complex enough to describe basic arithmetic propositions[18] especially when the consistency of the propositions are not assured by the axioms. For Godel, formal systems are not enough to exclusively proof certain mathematical truths, hence he provides some propositions whose truth validity can be clear enough to one who is not formally trained in mathematical reasoning or formal systems.

The logical synthesis of Godel's completeness (1929) and incompleteness (1931) theorems brings forth clear boundary between semantic truth and syntactic provability of formal systems. Thus while first order logic is complete, other axiomatic arithmetic system is inherently incomplete. The first-order logic is provable when its validity is true on every possible structure. On the other hand, Godel's incompleteness theorem shows that in spite of the consistency of arithmetic system, there are sentences that might be true while applying some standard models but false in other non

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[13].Godel, K. (1931). *On Formally Undecidable Propositions of Principia Mathematica and Related System I*. 12.

[14] Smullyan, R. M. (1992). *Godel's Incompleteness Theorem,* 5

[15].Godel, K. (1931). *On Formally Undecidable Propositions of Principia Mathematica and Related System I*. 12

[16]. Godel, K. (1931). *On Formally Undecidable Propositions of Principia Mathematica and Related System I*. (London: Dove Publications), 12.

[17]. Godel, K. (1931). *On Formally Undecidable Propositions of Principia Mathematica and Related System I*. 173

[18]. Godel, K. (1931). *On Formally Undecidable Propositions of Principia Mathematica and Related System I*. 174

standard models and vice versa. Thus for the reason that these sentences are false in at least one model, they are not logically valid according to the completeness theorem, hence they cannot be provable. Thus incompleteness theorem is therefore a discovery that arithmetic has some extra truths that logic alone or formal systems might not grasp since they are not true in every possible alternative world.

An epistemological foundational element of Thomas Aquinas' anthropology and psychology opines that intellectual engagement could be possible in non corporal or spiritual beings such as Angels as well as in corporal beings such as humans[19]. Furthermore, Aquinas claims that human beings also have corporal or material compositions and spiritual compositions. While the material composition in humans is responsible for formal logic, the non corporal or spiritual composition deals intuitions. Thus formal systems, arithmetic axiomatic rules and Turning machine can be seen as a formal process[20] and human mind with his spiritual dimension has the capacity for intuitive intellectual activity. The two theorems of Godel seem to separate these two aspects of the human person, spiritual and material which aid both intuitive and formal logic respectively. While the completeness theorem is interested in the formal logic formula if and only if it is logically valid in every possible structure, incompleteness theorem shows that there are sentences which are validity are assured by standard models but might be false in non standard models and vice versa. Thus for wholeness in knowing, this paper proposes a synthesis of complementarity between the two theorems; completeness and incompleteness. This is replicate of the synthesis inherent in the human person.

Conclusion

The works of Kurt Godel posed a great challenged the the popular claims of some very giant philosophers of the twentieth century such as Hilbert, Russell and Frege who constructed a tradition that exalted or "automate" mathematics in the modern time. This was a claim that arithmetic facts could be proven only by facts of consistency and completeness. This ambition is dashed by Godel's theorems and their hopes crushed. Godel shows not minding the strength and claims of mathematical theories, there are always as many mathematical facts it cannot prove, hence its own consistency is in doubt.

Following the studies of Aristotle and Aquinas in the area of metaphysics especially regarding the principle of causality, *agere sequitur esse;* "actions/doing is according to being" or "being act according to their kind". Thus having established that not minding the strength of mathematical theories, there are always as many mathematical facts it cannot prove, such could be said to exceed the capacity of physical laws of mathematics. Thus there ought to be another subject or being whose kind exceeds the capacity or potentials of the physical laws and who is able to act according to its own kind. These beigns are human beings whose intellectual ability goes beyond the limits of strict logicality and formal argumentation to intuitive reasoning or the *theoria*[21]. By logical argumentation, we mean formal syllogism, deductive or inductive reasoning capacity while the intuitive or *theoria*[22] refers to insights, creativity, imaginations, etc.

These two perspectives of the intellectual activity of the humans; strict logical argumentation and intuition provide strong leeway for Godel in his argument on incompleteness theorem. Following the studies of Godel, it behooves on the philosopher to establish strong ontological base of human beings which permits them to engage in intellectual activities which are not tenable to machines, mathematics, etc. Thus this paper concludes with the synthesis of complementarity of both the completeness and incompleteness theorems of Godel.

[19] Aquina, Summa Theologiae, Part 1, Question 75, Article 2 & 79.
[20] Turning , A. (1937). On Computer Numbers with an Application. (London: University Press), 544.
[21] Driessen, A. (1989). *Philosophical Consequences of the Godel Theorem,* 23.
[22] Driessen, A. (1989). *Philosophical Consequences of the Godel Theorem,* 23.