Nat. Volatiles & Essent. Oils, 2021; 8(4): 9870-9877

Some Fixed Point Theorem For Γ- Contraction In A Complete Partial B-Metric Spaces

Dr. A. Jennie Sebasty Pritha[1] , Dr. A. Mary Priya Dharsini[2] , Dr. A. Leema Maria Prakasam[3]

1, 2, 3 – Assistant Professor, PG and Research Department of Mathematics, Holy Cross College(Autonomous), Tiruchirappalli-620002. Mail id: jennipretha@gmail.com

Abstract

Matthews introduced the concepts of partial metric spaces and proved the Banach fixed point theorem in complete partial metric spaces.

Dukic, Kadelburg, and Radenovic proved fixed point theorems for Geraghty-type mappings in complete partial metric spaces.

Chang IL Kim and Giljun Han proved fixed point for some contractive mapping in a complete partial metric space.

In this paper, we generalize the concept of Chang and Giljun et.al. in the space of partial b-metric space.

Key words: Fixed point, contraction, partial b- metric space.

1. Introduction and Preliminaries

Metric spaces has been generalized in many ways. Among others, thenotion of a partial metric space was introduced in 1992 by Matthews [5]to model computation over a metric space. His goal was to study thereality of finding closer and closer approximation to a given number andshowing that contractive algorithms would serve to find these approximations.

In the sequel Bhaktin[10] and Czerwick[11] introduced b-metric spaces as a generalization of metric spaces. In 2013, Shukla[9] introduced the concept of partial b-metric spaces.

Mathematics Subject Classification: 47H10, 54H25.

Definition 1.1. Let X be a non-empty set. Then a mapping d:X x X →[0, ∞) is called a partial metric if for any x, y, z ∈ X, the following conditions hold:

(i) d(x, x) ≤ d(x, y),

(ii) d(x, y) = d(y, x),

(iii) if d(x, x) = d(x, y) = d(y, y), then x = y and

(iv) d(x, z) ≤ d(x, y) + d(y, z)- d(y, y)

In this case, (X, d) is called a partial metric space.

Definition 2.1[9] A partial b-metric on a non-empty set X is a mapping $\rho b : X \times X \to \mathbb{R}^+$ such that for all $x, y, z \in X$:

(i) $x = y$ if and only if $\rho_b(x,x) = \rho_b(x,y) = \rho_b(y,y)$;

(ii) $\rho_b(x,x) \le \rho_b(x,y)$;

(iii) $\rho_b(x,y) = \rho_b(y,x)$;

(iv) there exists a real number s ≥ 1 such that

$$\rho_b(x,y) \le s\left[\rho_b(x,z) + \rho_b(z,y)\right] - \rho_b(z,z).$$

A partial b-metric space is a pair (X, ρ_b) such that X is a non-empty set and ρ_b is a partial b-metric on X. The number s is called the coefficient of (X, ρ_b).

Proposition 2.1:[12]

Every partial b-metric $\rho_b$defines a b-metric d $_{\rho_b}$, where

$$d_{\rho_b}(x,y) = 2\rho_b(x,y) - \rho_b(x,x) - \rho_b(y,y) \text{ for all } x, y \in X.$$

Lemma 2.1[12]

(i) A sequence $\{x_n\}$ is a ρ_b -Cauchy sequence in a partial b-metric space (X, ρ_b) if and only if it is a b-Cauchy sequence in the b-metric space (X, d_{ρ_b}).

(ii) A partial b-metric space (X, ρ_b) is ρ_b -comlete is and only if the b-metric space (X, d ρ_b) is b-complete. Moreover, $\lim_{n\to\infty} d_{\rho b}(x, x_n) = 0$ if and only if

$$\lim_{n\to\infty} \rho_b(x, x_n) = \lim_{n,m\to\infty} \rho_b(x_n, x_m) = \rho_b(x,x).$$

Lemma 1.2. [6] Let (X, d) be a partial metric space. Then we have the followings:

(1) $\{B_d(x, \varepsilon) | x \in X, \in > 0\}$ is a base for some topology τ_d,

(2) (X, τ_d) is a T_0-space and

(3) a sequence $\{x_n\}$converges to x in (X, τ_d) if and only if $\lim_{n\to\infty} d(x_n, x) = d(x, x)$

Let (X, d) be a partial metric space. A sequence $\{x_n\}$ in (X, d) is called Cauchy if $\lim_{n\to\infty} d(x_m, x_n)$ exists and is finite and (X, d) is called complete if every Cauchy sequence $\{x_n\}$ in (X, d) converges to x in (X, τ_d) such that $\lim_{n\to\infty} d(x_n, x) = d(x, x) = \lim_{n\to\infty} d(x_m, x_n)$

Lemma 1.3. [8] Let (X, d) be a partial metric space. Then a sequence $\{x_n\}$ converges to x in (X,τ_d) with d(x; x) = 0 if and only if for any y ∈ X, $\lim_{n\to\infty}$ d(x_n, y) = d(x, y).

There exist many generalizations of the well-known Banach contraction mapping principle in the literature. In particular, Matthews [5],[6] proved the Banach fixed point theorem in partial metric spaces and after that, fixed point results in partial metric spaces have been studied by many authors([1], [3], [7]).

First, the well-known Banach contraction theorem [2] is stated as follows.

2. **Fixed point theorem for some contractive mapping in a complete partial b-metric space.**

In this section, we will prove a fixed point theorem for some contraction in complete partial b-metric spaces.

THEOREM2.1

Let (X, p) be a complete partial b-metric spaces and $f: X \to X$ a mapping such that there is a γ∈ ∑ such that

$p(f(u), f(v)) \leq s\, \gamma(B(u, v))\, B(u, v)$ (1)

for all $u, v \subset X$,where

$$A(u, v) = \max\{sp(u.v), sp(u, f(u)), sp(v, fv), {}^{s}/_{2}\, [p(u, fv) + p(fu, v)]\}$$

Then there is a unique fixed point‘a’ of f with $p\,(a, a) = 0$

Proof:

Let $u \in X$ and for any $n \in N$,

Let$f^{n+1}(u)$ = $f(f^n u)$and$u_n = f^n u$

Suppose that $p(u_m, u_{m+1}) = 0$for some $m \in N$

Hence one has the results.

Suppose that$p(u_n, u_{n+1}) > 0$ for all $n \in N$

For any $n \in N$, let$\alpha_n = p(u_n, u_{n+1})$ (1)

$p(u_n, u_n) \le \gamma(B(u_{n-1}, u_{n-1}))\ B(u_{n-1}, u_{n-1}) \le \gamma(B(u_{n-1}, u_{n-1}))\ \alpha_{n-1}$

for all $n \in N$. Then by (1),

$\alpha_{n+1} \le s\ \gamma(B(u_{n+1}, u_n))\ B(u_{n+1}, u_n)$ (2)

$\le s\ \gamma(B(u_{n+1}, u_n)) \max\{s\ \alpha_n,\ s\ \alpha_{n+1}, {}^s/_2[p(u_{n+1}, u_{n+1}) + p(u_{n+2}, u_n)]\}$,

for all $n \in N$.

If $\max\{s\ \alpha_n,\ s\ \alpha_{n+1}, {}^s/_2[p(u_{n+1}, u_{n+1}) + p(u_{n+2}, u_n)]\} = s\ \alpha_{n+1}$

for some $n \in N$. Then by (2), we have $\alpha_{n+1} = 0$, because $\alpha_{n+1} \neq 0$ and

$0 \le s\ \gamma(B(u_{n+1}, u_n)) < 1$, which is a contradiction.

$\alpha_{n+1} < \max\{s\ \alpha_n,\ s\ \alpha_{n+1}, {}^s/_2[p(u_{n+1}, u_{n+1}) + p(u_{n+2}, u_n)]\}$ (3)

for all $n \in N$. Hence by (2.2) and (2.3), we get

$\alpha_{n+1} \le s\ \gamma(B(u_{n+1}, u_n)) \max\{s\ \alpha_n, {}^s/_2[p(u_{n+1}, u_{n+1}) + p(u_{n+2}, u_n)]\}$ (4)

Suppose that there is an $n \in N$ such that

$\alpha_n \le {}^s/_2\ [\ p\ (u_{n+1}, u_{n+2}) + p\ (u_{n+2}, u_n) - p\ (u_{n+2}, u_{n+2})\ ]$ (5)

$\le {}^s/_2\ \alpha_{n+1} + {}^s/_2\ \alpha_n - {}^s/_2\ \alpha_{n+2}$ (6)

$\alpha_n \le s\ \alpha_{n+1}$ (7)

By (6) and (7), we have

${}^s/_2\ [p(u_{n+1}, u_{n+2}) + p(u_{n+2}, u_n) - p(u_{n+2}, u_n)] \le {}^s/_2\ \alpha_{n+1} + {}^s/_2\ \alpha_n$

$\le s\ \alpha_{n+1} - {}^s/_2\ \alpha_{n+2}$

Thus

α_{n+1} = max { α_n, $^1/_2$[p (u_{n+1}, u_{n+2}) + p(u_{n+2}, u_n) - p(u_{n+2}, u_n)]}

which is a contradiction. Hence we have

$$\alpha_n > {}^1/_2[\, p\,(u_{n+1}, u_{n+2}) + p(u_{n+2}, u_n)] \text{ for all } n \in N \qquad (8)$$

Thus by (4) & (5), we have

$\alpha_{n+1} \leq \gamma(\, B\,(u_{n+1}, u_n))\, \alpha_n < \alpha_n$ for all n ∈ N.

So $\{\alpha_n\}$ is a bounded below real decreasing sequence.

Thus there is a non-negative real number α with $\lim_{n\to\infty} \alpha_n = \alpha$.

Suppose that α> 0. Letting n →∞ in (9), we get

$\lim_{n\to\infty} \gamma(\, B\,(u_{n+1}, u_n)) = 1$

Since $\gamma \in \sum$ and $B\,(u_{n+1}, u_n)) = \alpha_n$

$0 = \lim_{n\to\infty} B\,(u_{n+1}, u_n) = \lim_{n\to\infty} \alpha_n \quad = \alpha$,

which is a contradiction.

Thus $\alpha = 0$ and so

$$\lim_{n\to\infty} p\,(u_n, u_{n+1}) = 0 \qquad (10)$$

To prove: $\{\alpha_n\}$ is a Cauchy sequence in (X, p).

It is enough to show that $\lim_{n,m\to\infty} p\,(u_m, u_n) = 0$.

Suppose that $\lim_{n,m\to\infty} p\,(u_m, u_n) \neq 0$. Then there is an ϵ> 0 and there are subsequences $\{u_{m(l)}\}$, $\{u_{n(l)}\}$of $\{u_n\}$ such that m(l) > n(l) >l and

$$p\,(u_{m(l)}, u_{n(l)}) \geq \epsilon \quad (11)$$

Moreover, for any $l \in N$, we can choose $m(l)$ in such a way that it is smallest integer with $m(l) > n(l)$ and satisfies (11).

Then$p\,(u_{m(l)-1}, u_{n(l)}) < \epsilon$ (12)

By (11) and (12), we have

$\epsilon \leq p\ (u_{m(l)}, u_{n(l)})$

$\leq s[p\ (u_{m(l)}, u_{m(l)-1}) + p\ (u_{m(l)-1}, u_{n(l)}) - p\ (u_{m(l)-1}, u_{m(l)-1})]$

$\leq s\ [p\ (u_{m(l)}, u_{m(l)-1}) + p\ (u_{m(l)-1}, u_{n(l)})]$

$< s\ p\ (u_{m(l)}, u_{m(l)-1}) + s\ \epsilon$ (13)

for all $l \in N$.

By (10) and (13), we have

$\epsilon \leq p\ (u_{m(l)}, u_{n(l)}) < s\ \epsilon$

$\lim_{l\to\infty} p\ (u_{m(l)}, u_{n(l)}) = \epsilon$ (14)

and

$p\ (u_{m(l)-1}, u_{n(l)-1}) \leq s\{\ p\ (u_{m(l)-1}, u_{m(l)}) + p\ (u_{m(l)}, u_{n(l)})] +$

$p\ (u_{n(l)}, u_{n(l)-1})]$

for all $l \in N$.

Letting $l \to \infty$ in the last inequality, by (10), we get

$\lim_{l\to\infty} p\ (u_{m(l)-1}, u_{n(l)-1}) \leq \epsilon$

Since

$p\ (u_{m(l)}, u_{n(l)}) \leq s\ [\ p\ (u_{m(l)}, u_{m(l)-1}) + p\ (u_{m(l)-1}, u_{n(l)-1}) +$

$p\ (u_{n(l)-1}, u_{n(l)})]$

for all $l \in N$, $\epsilon \leq s \lim_{l\to\infty} p\ (u_{m(l)-1}, u_{n(l)-1})$ (16)

By (15) and (16), we have $\epsilon \leq s \lim_{l\to\infty} p\ (u_{m(l)-1}, u_{n(l)-1}) \leq \epsilon$

$p\ (u_{m(l)-1}, u_{n(l)-1}) = \epsilon$ (17)

By (10) and (12), we have

$\epsilon \leq p\,(u_{m(l)}, u_{n(l)} \leq s\gamma(B(u_{m(l)-1}, u_{n(l)-1}))\, B(u_{m(l)-1}, u_{n(l)-1})$

$\leq s\,\gamma(B(u_{m(l)-1}, u_{n(l)-1}))\, \max\{ s\, p(u_{m(l)-1}, u_{n(l)-1}), sp(u_{m(l)-1}, u_{m(l)}),$

$sp(u_{n(l)}, u_{n(l)-1}),\ {}^{s}/_{2}\, [\, p\,(u_{m(l)-1}, u_{n(l)}) + p\,(u_{m(l)}, u_{n(l)-1}]\}$

$\leq \gamma(B(u_{m(l)-1}, u_{n(l)-1}))\, \max\{ s\, p(u_{m(l)-1}, u_{n(l)-1}), s\, p(u_{m(l)-1}, u_{m(l)}),$

$sp(u_{n(l)}, u_{n(l)-1}),\ {}^{s}/_{2}\, [\epsilon + p\,(u_{m(l)}, u_{n(l)}) + p\,(u_{n(l)}, u_{n(l)-1}]\}$

By (14) and (17), we get$\lim_{l\to\infty} s\, p(B\,(u_{m(l)-1}, u_{n(l)-1})) = 1$

Hence $\lim_{l\to\infty} B\,(u_{m(l)-1}, u_{n(l)-1})) = 0$

and so$\lim_{l\to\infty} p\,(u_{m(l)}, u_{n(l)}) = 0$

which is a contradiction to (11). Thus

$\lim_{m,n\to\infty} p\,(u_m, u_n) = 0$

and so $\{u_n\}$ is a Cauchy sequence in (X, p). Since (X, p) is a completepartial metric space, there is 'a' $\in$ X such that

$\lim_{n\to\infty} p(u, a) = p(a, a) = \lim_{m,n\to\infty} p\,(u_m, u_n)$

References

[1] I. Altun and K. Sadarangani, Generalized Geraghty type mappings on partialmetric spaces and fxed point results, Arab J. Math., 2 (2013), 247{253.

[2] S. Banach, Surles operation dans les ensembles abstraits et leur application auxequation integrals, Fundamenta Mathematicae, 3 (1922), 133{181.

[3] M. Bukatin, R. Kopperman, S. Matthews, and H. Pajoohesh, Partial metricspaces, American Mathematical Monthly, 116 (2009), 708-718.

[4] D. Dukic, Z. Kadelburg, Z. and S. Radenovic, Fixed points of Geraghty-typemappings in various generalized metric spaces, Abstract and Applied Analysis.,2 (2011), 1-13.

[5] S. G. Matthews, Partial metric topology, Research Report 212, Dep. of Computer Science, University of Warwick, 1992.

[6] S. G. Matthews, Partial metric topology, Proc. 8th Summer Conference on General Topology and Applications, Ann. New York Acad. Sci. 728 (1994), 183{197.

[7] S. Oltra, and O. Valero, Banach's fixed theorem for partial metric spaces, Rend.Istit. Mat.

Univ. Trieste, 36 (2004) 17-26.
[8] D. Paesano and P. Vetro, Suzuki's type characterizations of completeness for partial metric spaces and fixed points for partially ordered metric spaces, TopologyAppl., 159 (2012), 911 - 920.
[9]Shukla.S, “Partial b-metric spaces and fixed point theorems”, Mediterr. J. Math. vol.11 (2014), 703-711.
[10] Bakhtin I.A, “The contraction mapping principle in quasi metric spaces”. Funct. Anal. Unianowsk Gos. Ped. Inst. Vol. 30, 26-37 (1989).
[11]Czerwik. S, “Contraction mappings in b-metric spaces”, Acta Mathematica et Informatica Universitatis Ostraviensis Vol. 5-11 (1993).