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Numerical Solution Of Nonlinear Fractional Differential Equations Using Kharrat-Toma Iterative Method

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Abstract: In this paper, the Kharrat-Toma Transform of Riemann-Liouville and Caputo fractional integral and derivatives are derived. Also, Kharrat-Toma Iterative Method is proposed to find the numerical solution of the nonlinear fractional differential equations. This method is a combination of Kharrat-Toma Transform and Iterative method. The results of illustrative examples reveal that our proposed method is effective and powerful to solve the nonlinear fractional differential equations. A comparison is made with an exact solution and existing method.

Keywords: Fractional Differential Equations, Caputo derivative, Riemann-Liouville derivative, Kharrat-Toma Transform, Kharrat-Toma Iterative method.

I Introduction

Fractional calculus is a generalization of ordinary differentiation and integration to arbitrary non-integer order. Nowadays, an increasing attention is paid by the researches in the development of fractional differential equations (FDEs) in several fields such as mechanics, thermal systems, image processing, viscoelastic and fluid flow [1-3]. Various analytical and numerical methods are proposed to solve the linear and nonlinear FDEs. Some of them are Homotopy Analysis Method [4], Adomian Decomposition Method [5] and Differential Transform Method [6].

Daftardar-Gejji and Jafari [7] introduced the iterative method for obtaining the numerical solution of nonlinear functional equations. Iterative method is one of the important method to solve nonlinear differential equations. Jafari et al. [8] developed Iterative Laplace Transform method to obtain the numerical solution of a system of fractional partial differential equations. Later, many researches applied several transforms such as Elzaki [9], Sumudu [10] and Mahgoub [11] in this iterative method to obtain the numerical solution of fractional partial differential equations. Kharrat and Toma [12] introduced anew transform called Kharrat-Toma Transform to solve the ordinary differential equations with initial conditions. Toma and Alturky [13] combined a Hybrid Kharrat-

Toma transform with Homotopy Perturbation method for finding the exact and approximate solutions of linear and nonlinearIntegro-differential equations.

In this paper, Kharrat-Toma Iterative Method (KTIM) is developed for solving fractional differential equation numerically. The paper is organized as follows: Section 2 gives the fundamental definitions and properties of fractional calculus. Section 3 contains the KharratToma Transform of fractional integrals and derivatives. Section 4 delivers the construction of KTIM. Section 5 presents some numerical examples of FDEs to show the effectiveness of the proposed method by means of some comparison with exact solution and MahgoubAdomian Decomposition Method (MADM) [14].

II Preliminaries and Notations

In this section, some basic definitions and properties of fractional calculus are given.

Definition 1: A real function $f(t), t > 0$ is said to be in the space $C_\mu,\ \mu \in \mathbb{R}$if there exists a real number $p > \mu$ such that$f(t) = t^p f_1(t)$ where $f_1(t) \in C[0,\infty)$ and it is said to be in the spaceC_μ^n if and only if $f^{(n)} \in C_\mu,\ n \in \mathbb{N}$. [2]

Definition 2: Riemann Liouville fractional integral operatorI_t^α of order $\alpha \in R, \alpha > 0$ of function$f(t) \in C_\mu, \mu \geq -1$ is defined as [15],

$$I_t^\alpha f(t) = D_t^{-\alpha} f(t) = \frac{1}{\Gamma\alpha}\int_0^t (t-\tau)^{\alpha-1} f(\tau)d\tau, \qquad t > 0 \qquad (1)$$

$$I_t^0 f(t) = f(t)$$

where$\Gamma(.)$ denotes the Gamma function.

Definition 3: Riemann-Liouville fractional derivative${}^{RL}D_t^\alpha f(t)$ of order $\alpha \in R,\ \alpha > 0$ of function$f(t) \in C_\mu,\ \mu \geq -1$ is defined as [16],

$${}^{RL}D_t^\alpha f(t) = \left(\frac{d}{dt}\right)^n \left(I_t^{n-\alpha} f(t)\right)$$

$$= \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \int_0^t \frac{f(\tau)d\tau}{(t-\tau)^{\alpha-n+1}},\ t > 0, \qquad n-1 < \alpha < n \qquad (2)$$

Definition 4: Caputo fractional derivative of order $\alpha \in R,\ \alpha > 0$ of function $f(t) \in C_\mu$, $\mu \geq -1$is given by [16],

$${}^{c}D_t^\alpha f(t) = I_t^{n-\alpha} D^n f(t) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)}\displaystyle\int_0^t (t-\tau)^{n-\alpha-1} f^n(\tau)d\tau, & n-1 < \alpha < n, n \in \mathbb{N} \\ \dfrac{d^n f(t)}{dt^n}, & \alpha = n \end{cases} \qquad (3)$$

III Kharrat-Toma Transform of Fractional Integrals and Derivatives

Kharrat-Toma Transform of the function $f(t)$ for $t \geq 0$ is defined by the integral

$$B[f(t)] = G(s) = s^3 \int_0^\infty f(t) e^{\frac{-t}{s^2}} dt, \quad t \geq 0, \tag{4}$$

and it is denoted by the operator $B(.)$

If $B[f(t)] = G(s)$, then$f(t)$ is called the inverse Kharrat-Toma Transform of $G(s)$.In symbol,

$$f(t) = B^{-1}[G(s)] = B^{-1}\left[s^3 \int_0^\infty f(t) e^{\frac{-t}{s^2}} dt\right]$$

whereB^{-1} is called the inverse Kharrat-Toma Transform operator.

Kharrat-Toma Transform of simple functions are given below:

(i)$B[1] = s^5$

(ii) $B[t^n] = n!\, s^{2n+5} = \Gamma(n+1) s^{2n+5}, \; n \geq 0$ (5)

(iii) $B[\sin at] = \dfrac{as^7}{1 + a^2 s^4}$

(iv) $B[\cos at] = \dfrac{s^5}{1 + a^2 s^4}$

(v) $B[\sinh at] = \dfrac{as^7}{1 - a^2 s^4}$

(vi) $B[\cosh at] = \dfrac{s^5}{1 - a^2 s^4}$

Some basic properties of the Kharrat-Toma Transform are given as follows:

Property 1: The Kharrat-Toma Transform is a linear operator. That is, if $c_1, c_2, \ldots, c_n$are non-zero constants, then

$$B\left[\sum_{i=1}^{n} c_i\, f_i(t)\right] = \sum_{i=1}^{n} c_i\, B[f_i(t)]$$

Property 2: Let $n \geq 1$ and $G(s)$ be the Kharrat-Toma Transform of the function $f(t)$. The Kharrat-Toma Transform of n^{th} derivative of $f(t)$ is given by

$$B[f^n(t)] = \frac{1}{s^{2n}} G(s) - \sum_{k=0}^{n-1} s^{-2n+2k+5} f^k(0) \tag{6}$$

Property 3: Let $M(s)$ and $N(s)$ denote the Kharrat-Toma Transform of $f(t)$ and $g(t)$ respectively. If

$$(f * g)(t) = \int_0^t f(\tau)\, g(t - \tau) d\tau \tag{7}$$

where$*$ denotes convolution of f and g, then the Kharrat-Toma Transform of the convolution of $f(t)$ and $g(t)$ is

$$B[f(t) * g(t)] = \frac{1}{s^3} M(s) N(s)$$

Some fundamental properties of Kharrat-Toma Transformnecessary in solving FDEs are given in the following theorems.

Theorem 1: Let $n \in \mathbb{N}$ and $\alpha > 0$ be such that $n-1 \leq \alpha < n$ and $G(s)$ be the Kharrat-Toma Transformof $f(t)$, then Kharrat-Toma Transform of Riemann-Liouville fractional integral of $f(t)$ of order α is given by

$$B[I^{\alpha}f(t)] = B\left[{}^{RL}D_t^{-\alpha}f(t)\right] = s^{2\alpha}G(s)$$

Proof:

We know that the definition of Riemann-Liouville fractional integral is of the form

$$I^{\alpha}f(t) = {}^{RL}D_t^{-\alpha}f(t) = \frac{1}{\Gamma(\alpha)}\int_{\mathbf{0}}^{\mathbf{t}}(t-\tau)^{\alpha-1}f(\tau)d\tau$$

$$= \frac{1}{\Gamma(\alpha)}[t^{\alpha-1} * f(t)] \tag{8}$$

Applying the Kharrat-TomaTransform in Eqn. (8), we obtain

$$B[I^{\alpha}f(t)] = \frac{1}{\Gamma(\alpha)}\frac{1}{s^3}B[t^{\alpha-1}]B[f(t)]$$

$$= \frac{1}{\Gamma(\alpha)}\frac{1}{s^3}[\Gamma(\alpha)s^{2\alpha+3}]G(s)$$

$$B[I^{\alpha}f(t)] = B\left[{}^{RL}D_t^{-\alpha}f(t)\right] = s^{2\alpha}G(s) \tag{9}$$

This completes the proof.

Theorem 2: Let $n \in \mathbb{N}$ and $\alpha > 0$ be such that $n-1 < \alpha \leq n$ and $G(s)$ be the Kharrat-Toma Transform of the function $f(t)$, then theKharrat-TomaTransform of Riemann-Liouville fractional derivative of $f(t)$ of order α is given by

$$B\left[{}^{RL}D_t^{\alpha}f(t)\right] = s^{-2\alpha}G(s) - \sum_{k=0}^{n-1} s^{-2k+3}\left[\ D_t^{\alpha-k-1}f(t)\right]_{t=0} \tag{10}$$

Proof:

Let $\qquad D_t^{\alpha}f(t) = g^{(n)}(t) = \frac{d^n}{dt^n}g(t)$

Then,

$$g(t) = \frac{d^{-n}}{dt^{-n}}\frac{d^n}{dt^n}g(t)$$

$$= \frac{d^{-n}}{dt^{-n}}\ D_t^{\alpha}f(t)$$

$$g(t) = \ D_t^{-(n-\alpha)}f(t) \tag{11}$$

Applying Kharrat-TomaTransform on both sides of Eqn. (11) and by using Theorem 1, we get

$$H(s) = B[g(t)] = B\left[\ D_t^{-(n-\alpha)}f(t)\right] = s^{2n-2\alpha}G(s) \tag{12}$$

Also from Eqn. (6), we have

$$B\left[{}^{RL}D_t^{\alpha} f(t)\right] = B\left[\frac{d^n}{dt^n} g(t)\right]$$

$$= \frac{1}{s^{2n}} H(s) - \sum_{k=0}^{n-1} s^{-2k+3} g^{n-k-1}(t)\,|_{t=0} \tag{13}$$

From the definition of Riemann-Liouville fractional derivative, we obtain

$$g^{(n-k-1)}(t) = \frac{d^{n-k-1}}{dt^{n-k-1}} g(t)|_{t=0}$$

$$= \frac{d^{n-k-1}}{dt^{n-k-1}} \quad D_t^{-(n-\alpha)} f(t)|_{t=0}$$

$$= D_t^{n-k-1}\left[\quad D_t^{-(n-\alpha)} f(t)|_{t=0}\right]$$

$$= \quad D_t^{\alpha-k-1} f(t)|_{t=0} \tag{14}$$

Hence, by substituting the Eqns. (12) and (14) in Eqn. (13), we get

$$B\left[\,{}^{RL}D_t^{\alpha} f(t)\right] = s^{-2\alpha} G(s) - \sum_{k=0}^{n-1} s^{-2k+3} \quad D_t^{\alpha-k-1} f(t)|_{t=0}\; n \in \mathbb{N}$$

Theorem 3: Let $n \in \mathbb{N}$ and $\alpha > 0$ be such that $n - 1 < \alpha \leq n$ and $G(s)$ be the Kharrat-TomaTransform of the function $f(t)$, then the Kharrat-Toma Transform of Caputo fractional derivative of $f(t)$ of order α is given by

$$B[\,{}^{c}D_t^{\alpha} f(t)] = s^{-2\alpha} G(s) - \sum_{k=0}^{n-1} s^{2k-2\alpha+5} f^{k}(t)|_{t=0}\; n - 1 < \alpha \leq n,\;\; n \in \mathbb{N} \tag{15}$$

Proof

Let $g(t) = f^{(n)}(t)$ then, by the definition of Caputo fractional derivative we obtain

$${}^{c}D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^{n}(\tau) d\tau, \qquad t > 0$$

$$= \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} g(\tau) d\tau$$

$$= \quad D_t^{-(n-\alpha)} g(t) \tag{16}$$

Applying the Kharrat-Toma Transform on both sides of Eqn. (16) and using Theorem 1we get,

$$B[\,{}^{c}D_t^{\alpha} f(t)] = B\left[\quad D_t^{-(n-\alpha)} g(t)\right] = s^{2n-2\alpha} H(s) \tag{17}$$

Also, we have $B[g(t)] = B\left[f^{(n)}(t)\right]$

$$H(s) = \frac{1}{s^{2n}} G(s) - \sum_{k=0}^{n-1} s^{-2n+2k+5} f^{k}(t)|_{t=0}$$

Hence, Eqn. (17) becomes,

$$B[\,{}^{c}D_t^{\alpha}f(t)] = s^{2n-2\alpha}\left[\frac{1}{s^{2n}}G(s) - \sum_{k=0}^{n-1} s^{-2n+2k+5} f^{k}(t)|_{t=0}\right],\ \ n-1<\alpha<n$$

$$= s^{-2\alpha}G(s) - \sum_{k=0}^{n-1} s^{2k-2\alpha+5} f^{k}(t)|_{t=0}$$

This completes the proof.

IV. Construction of KTIM for Solving Fractional Differential Equation

Consider the following nonlinear fractional differential equation

$$ {}^{c}D^{\alpha}y(t) + Ry(t) + Ny(t) = f(t), t \geq 0, \text{ for } n-1<\alpha\leq n\,, n\ \in \mathbb{N} \tag{18}$$

subject to the initial condition

$$y^{(k)}(0) = b_k, \tag{19}$$

whereb_k are known real constants. ${}^{C}D^{\alpha}y(t)$denotes the fractional order derivative in Caputo sense. R is a linear operator. N is a nonlinear operator and $f(t)$ is a known function. Let $[0, T]$ be the interval over which we need to find the solution of the above initial value problem.

Applying the Kharrat-TomaTransform to both sides of Eqn. (18) and by using the linearity of Kharrat-Toma Transform, the result is

$$B(\,{}^{c}D^{\alpha}y(t)) + B\left(R\big(y(t)\big)\right) + B\left(N\big(y(t)\big)\right) = B(f(t))$$

Using Theorem 3 in the above equation, we get

$$s^{-2\alpha}B\big(y(t)\big) = \sum_{k=0}^{n-1} s^{2k-2\alpha+5}y^{(k)}(0) + B\big(f(t)\big) - B\left(R\big(y(t)\big)\right) - B\left(N\big(y(t)\big)\right)$$

$$B\big(y(t)\big) = \frac{1}{s^{-2\alpha}}\sum_{k=0}^{n-1} s^{2k-2\alpha+5}y^{(k)}(0) + \frac{1}{s^{-2\alpha}}B\big(f(t)\big) - \frac{1}{s^{-2\alpha}}B\left(R\big(y(t)\big)\right) - \frac{1}{s^{-2\alpha}}B\left(N\big(y(t)\big)\right) \tag{20}$$

The KTIM represents the solution as an infinite series

$$y(t) = \sum_{n=0}^{\infty} y_n(t) \tag{21}$$

Now, the linear operator R becomes,

$$R\left(\sum_{n=0}^{\infty} y_n(t)\right) = \sum_{n=0}^{\infty} R(y_n(t)) \tag{22}$$

and the nonlinear term $N\big(y(t)\big)$ is decomposed as

$$N\left(\sum_{n=0}^{\infty} y_n(t)\right) = N(y_0(t)) + \sum_{n=1}^{\infty}\left\{N\left(\sum_{k=0}^{n} y_k(t)\right) - N\left(\sum_{k=0}^{n-1} y_k(t)\right)\right\} \tag{23}$$

Substituting Eqns. (21) - (23) in Eqn. (20), we have

$$B\left(\sum_{n=0}^{\infty} y_n(t)\right) = \frac{1}{s^{-2\alpha}}\left[\sum_{k=0}^{n-1} s^{2k-2\alpha+5} y^k(0) + B\big(f(t)\big)\right]$$

$$-\left[\frac{1}{s^{-2\alpha}} B\left[\sum_{n=0}^{\infty} R\big(y_n(t)\big) + N\big(y_0(t)\big) + \sum_{n=1}^{\infty}\left\{N\left(\sum_{k=0}^{n} y_k(t)\right) - N\left(\sum_{k=0}^{n-1} y_k(t)\right)\right\}\right]\right] \quad (24)$$

Hence the iterations are defined by the following recursive algorithm

$$B(y_0(t)) = \frac{1}{s^{-2\alpha}}\left[\sum_{k=0}^{n-1} s^{2k-2\alpha+5} y^k(0) + B\big(f(t)\big)\right] \quad (25)$$

$$B(y_1(t)) = -\frac{1}{s^{-2\alpha}} B\big[R\big(y_0(t)\big) + N(y_0)\big] \quad (26)$$

$$\vdots$$

$$B\big(y_n(t)\big) = -\frac{1}{s^{-2\alpha}} B\left[R\big(y_{n-1}(t)\big) + \left\{N\left(\sum_{k=0}^{n} y_k(t)\right) - N\left(\sum_{k=0}^{n-1} y_k(t)\right)\right\}\right], n \geq 1 \quad (27)$$

Using the initial conditions (19) and applying the inverse Kharrat-TomaTransform to equations (25)-(27) we obtain the values $y_0(t), y_1(t), y_2(t), \dots, y_n(t)$ recursively.

Therefore the n-term approximate solution is given by

$$y(t) = y_0(t) + y_1(t) + y_2(t) + \cdots + y_n(t), \qquad n = 1, 2, \dots$$

V Numerical Examples

Example 1

Consider the nonlinear fractional differential equation

$$ {}^{c}D^{\alpha} y(t) = y^2 + 1, \quad n-1 < \alpha \leq n, \; 0 < t \leq 1, \quad (28)$$

subject to the initial conditions

$$y^{(i)}(0) = 0, \qquad i = 0, 1, 2, \dots, m-1 \quad (29)$$

The exact solution of Eqn. (28) is$y(t) = \tan t$ when $\alpha = 1$

Applying the Kharrat-Toma Transform in the Eqn. (28), then

$$B({}^{c}D^{\alpha} y(t)) = B(y^2 + 1)$$

Using Theorem 3 and the initial conditions (29), then we have

$$B\big(y(t)\big) = \frac{1}{s^{-2\alpha}}(B(y^2)) + \frac{s^5}{s^{-2\alpha}} \quad (30)$$

Applying the inverse Kharrat-Toma Transform in Eqn. (30) we obtain

$$y(t) = B^{-1}\left[\frac{1}{s^{-2\alpha}}(B(y^2)) + \frac{s^5}{s^{-2\alpha}}\right]$$

In the view of the recurrence relations (25) – (27) we get

$$y_0(t) = B^{-1}\left[\frac{s^5}{s^{-2\alpha}}\right]$$

$$y_1(t) = B^{-1}\left[\frac{1}{s^{-2\alpha}}B(y_0^2)\right]$$

$$y_2(t) = B^{-1}\left[\frac{1}{s^{-2\alpha}}B((y_0 + y_1)^2)\right] - B^{-1}\left[\frac{1}{s^{-2\alpha}}B(y_0^2)\right]$$

$$y_3(t) = B^{-1}\left[\frac{1}{s^{-2\alpha}}B((y_0 + y_1 + y_2)^2)\right] - B^{-1}\left[\frac{1}{s^{-2\alpha}}B((y_0 + y_1)^2)\right]$$

$$\vdots$$

Therefore,

$$y_0(t) = \frac{t^\alpha}{\Gamma(\alpha+1)}$$

$$y_1(t) = \frac{\Gamma(2\alpha+1)}{\left(\Gamma(\alpha+1)\right)^2}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}$$

$$y_2(t) = \left(\frac{\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)}\right)^2\frac{\Gamma(6\alpha+1)}{\Gamma(7\alpha+1)}\frac{t^{7\alpha}}{\left(\Gamma(\alpha+1)\right)^4} + \frac{2\Gamma(2\alpha+1)\Gamma(4\alpha+1)}{\left(\Gamma(\alpha+1)\right)^3\Gamma(3\alpha+1)}\frac{t^{5\alpha}}{\Gamma(5\alpha+1)}$$

$$\vdots$$

The approximate solution is given by

$$y(t) = y_0 + y_1 + y_2 + y_3 + \cdots$$

$$\text{i.e.,}\quad y(t) = \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{\Gamma(2\alpha+1)}{\left(\Gamma(\alpha+1)\right)^2}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \left(\frac{\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)}\right)^2\frac{\Gamma(6\alpha+1)}{\Gamma(7\alpha+1)}\frac{t^{7\alpha}}{\left(\Gamma(\alpha+1)\right)^4} + \frac{2\Gamma(2\alpha+1)\Gamma(4\alpha+1)}{\left(\Gamma(\alpha+1)\right)^3\Gamma(3\alpha+1)}\frac{t^{5\alpha}}{\Gamma(5\alpha+1)} + \cdots$$

The following figure 1 is the graphical representation of exact solution and the numerical solution of Eqn. (28) using KTIM for $\alpha = 0.5, 1, 1.5$ and 2.5.

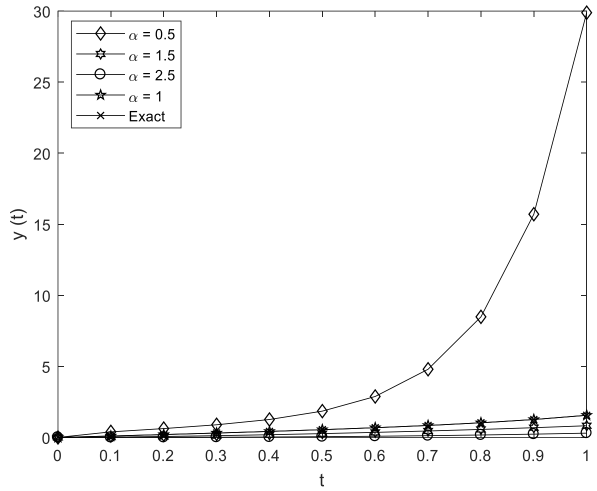

Fig. 1: Exact and numerical solutions of Eqn. (28)

Table 1 shows the solution of Example 1 for different values of α. The absolute error is calculated by finding the difference between the discrete values and the exact solutions for $0 < t \leq 1$. Comparison between KTIM and MADM of Eqn. (28) is shown in Table 1. KTIM gives the good result than MADM.

Table 1: Numerical Solution of Eqn. (28) using KTIM for different values of α

t	$\alpha = 1$				$\alpha = 0.5$	$\alpha = 1.5$	$\alpha = 2.5$
	MADM	Exact	KTIM	Absolute Error			
0.1	0.100335	0.100335	0.100335	1.553710E-14	0.391970	0.023790	0.000952
0.2	0.202710	0.202710	0.202710	3.269958E-11	0.623658	0.067330	0.005383
0.3	0.309336	0.309336	0.309336	2.963048E-09	0.890524	0.123896	0.014833
0.4	0.422793	0.422793	0.422793	7.497385E-08	1.261590	0.191362	0.030450
0.5	0.546302	0.546302	0.546302	9.528815E-07	1.851229	0.268856	0.053197
0.6	0.684135	0.684137	0.684129	7.912239E-06	2.881687	0.356238	0.083925
0.7	0.842269	0.842288	0.842239	4.946815E-05	4.797168	0.453950	0.123412
0.8	1.029511	1.029639	1.029586	5.253994E-05	8.485896	0.563007	0.172391
0.9	1.259443	1.260158	1.260087	7.135820E-05	15.706593	0.685056	0.231574
1.0	1.553901	1.557408	1.557313	9.462219E-05	29.888579	0.822511	0.301676

Example 2

Consider the nonlinear fractional differential equation

$$ {}^{c}D^{\alpha}y(t) = 2y - y^2 + 1, \quad 0 < \alpha \leq 1, \; 0 < t \leq 1, \tag{31}$$

Subject to the initial condition

$$y(0) = 0 \tag{32}$$

The Exact solution of Eqn. (31) for $\alpha = 1$ (ODE) is

$$y(t) = 1 + \sqrt{2}\tanh\left(\sqrt{2}t + \frac{1}{2}\ln\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)\right)$$

Applying the Kharrat-Toma Transform in the Eqn. (31), then

$$B\left({}^{c}D^{\alpha}y(t)\right) = B(2y - y^2 + 1)$$

Using Theorem 3 and the initial conditions (32), then we have

$$B\big(y(t)\big) = \frac{2}{s^{-2\alpha}}B(y(t)) - \frac{1}{s^{-2\alpha}}(B(y^2)) + \frac{s^5}{s^{-2\alpha}}$$

In the view of the recurrence relations (25) – (27) we get

$$y_0(t) = B^{-1}\left[\frac{s^5}{s^{-2\alpha}}\right]$$

$$y_1(t) = B^{-1}\left[\frac{1}{s^{-2\alpha}}\Big(B\big(2y_0(t)\big) - B(y_0^2)\Big)\right]$$

$$y_2(t) = B^{-1}\left[\frac{1}{s^{-2\alpha}}\Big(B\big(2y_1(t)\big) - \Big(B((y_0 + y_1)^2) - B(y_0^2)\Big)\Big)\right]$$

$$y_3(t) = B^{-1}\left[\frac{1}{s^{-2\alpha}}\Big(B\big(2y_2(t)\big) - \Big(B((y_0 + y_1 + y_2)^2) - B((y_0 + y_1)^2)\Big)\Big)\right]$$

$$\vdots$$

Hence,

$$y_0(t) = \frac{t^{\alpha}}{\Gamma(\alpha + 1)}$$

$$y_1(t) = \frac{2t^{2\alpha}}{\Gamma(2\alpha + 1)} - \frac{\Gamma(2\alpha + 1)}{\big(\Gamma(\alpha + 1)\big)^2}\frac{t^{3\alpha}}{\Gamma(3\alpha + 1)}$$

$$y_2(t) = \frac{4t^{3\alpha}}{\Gamma(3\alpha+1)} - \frac{t^{4\alpha}}{\Gamma(4\alpha+1)}\left[\frac{2\Gamma(2\alpha+1)}{(\Gamma(\alpha+1))^2} + \frac{4\Gamma(3\alpha+1)}{\Gamma(\alpha+1)\Gamma(2\alpha+1)}\right]$$
$$+\frac{t^{5\alpha}}{\Gamma(5\alpha+1)}\left[\frac{2\Gamma(2\alpha+1)\Gamma(4\alpha+1)}{\Gamma(3\alpha+1)(\Gamma(\alpha+1))^3} - \frac{4\Gamma(4\alpha+1)}{(\Gamma(2\alpha+1))^2}\right]$$
$$-\left(\frac{\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)}\right)^2 \frac{\Gamma(6\alpha+1)}{\Gamma(7\alpha+1)}\frac{t^{7\alpha}}{(\Gamma(\alpha+1))^4} + \frac{4\Gamma(5\alpha+1)}{(\Gamma(\alpha+1))^2\Gamma(3\alpha+1)}\frac{t^{6\alpha}}{\Gamma(6\alpha+1)}$$

$\vdots$

The approximate solution is

$$y(t) = \frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{2t^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{\Gamma(2\alpha+1)}{(\Gamma(\alpha+1))^2}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \frac{4t^{3\alpha}}{\Gamma(3\alpha+1)}$$
$$-\frac{t^{4\alpha}}{\Gamma(4\alpha+1)}\left[\frac{2\Gamma(2\alpha+1)}{(\Gamma(\alpha+1))^2} + \frac{4\Gamma(3\alpha+1)}{\Gamma(\alpha+1)\Gamma(2\alpha+1)}\right]$$
$$+\frac{t^{5\alpha}}{\Gamma(5\alpha+1)}\left[\frac{2\Gamma(2\alpha+1)\Gamma(4\alpha+1)}{\Gamma(3\alpha+1)(\Gamma(\alpha+1))^3} - \frac{4\Gamma(4\alpha+1)}{(\Gamma(2\alpha+1))^2}\right]$$
$$-\left(\frac{\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)}\right)^2 \frac{\Gamma(6\alpha+1)}{\Gamma(7\alpha+1)}\frac{t^{7\alpha}}{(\Gamma(\alpha+1))^4} + \frac{4\Gamma(5\alpha+1)}{(\Gamma(\alpha+1))^2\Gamma(3\alpha+1)}\frac{t^{6\alpha}}{\Gamma(6\alpha+1)} + \cdots$$

Table 2 shows the solution of Eqn. (31)using KTIM for different values of α when$0 < t \leq 1$. The absolute error is calculated by finding the difference between the discrete values and the exact solutions for $0 < t \leq 1$. Comparison between KTIM and MADM of Eqn. (31) is shown in Table 2. KTIM gives the good result than MADM.

Table 2: Numerical Solution of Eqn. (31) using KTIM for different values of α

t	$\alpha = 1$				$\alpha = 0.5$	$\alpha = 0.75$	$\alpha = 0.95$
	MADM	**Exact**	**KTIM**	**Absolute Error**			
0.1	0.110295	0.110295	0.110294	1.085150E-06	0.589477	0.245327	0.128802
0.2	0.241977	0.241977	0.241950	2.682335E-05	0.929513	0.474291	0.275674
0.3	0.395122	0.395105	0.394957	1.474432E-04	1.172488	0.708194	0.443322
0.4	0.567934	0.567812	0.567398	4.142629E-04	1.344449	0.936359	0.628750
0.5	0.756482	0.756014	0.755257	7.575056E-04	1.469506	1.147505	0.826323
0.6	0.954756	0.953566	0.952589	9.775567E-04	1.569284	1.333420	1.028554
0.7	1.155089	1.152949	1.152054	8.948357E-04	1.661190	1.490501	1.227099
0.8	1.348968	1.346364	1.345790	5.736713E-04	1.757469	1.620123	1.413969

0.9	1.528179	1.526911	1.526506	4.056813E-04	1.864987	1.728113	1.582803
1.0	1.676254	1.689498	1.688651	8.470841E-04	1.985521	1.823534	1.729998

The following figure 2 is the graphical representation of exact solution and the numerical solution of Eqn. (31) using KTIM for$\alpha = 0.5, 0.75, 0.95$ and 1.

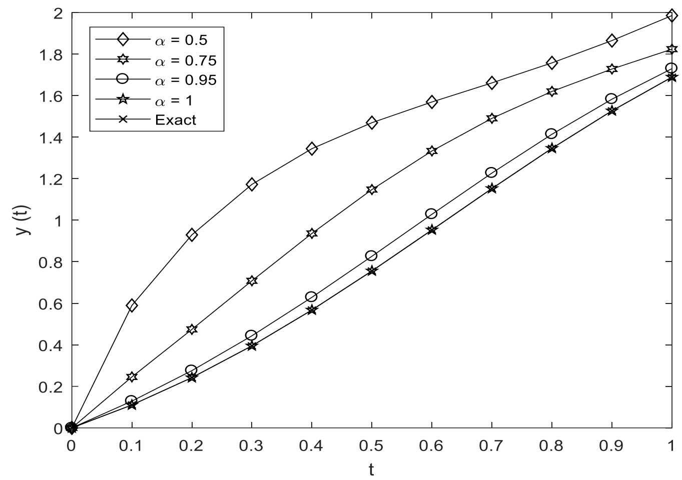

Fig. 2: Exact and numerical solutions of Eqn. (31)

Conclusion

In this paper, the Kharrat-Toma Transform of Riemann-Liouville and Caputo fractional integral and derivatives are proved in theorems. Also, the combined form of the Kharrat-Toma Transform and Iterative method is successfully applied to solve the fractional differential equations numerically. By comparing the numerical solution of FDEs using our proposed method with MADM, it can be concluded that KTIM gives good result than MADM. Therefore, the Kharrat-Toma Iterative Method is an efficient and effective method for finding numerical solutions of fractional differential equations.

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