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Common Fixed Point Theorems On D^{*}Metric Spaces

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Abstract

In this paper we have proved some fixed and common fixed point theorems in D metric spaces and extended it to prove the existence of a common fixed point onD^{*}METRIC SPACES.

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1. INTRODUCTION

In the 20th century, the theoretical foundation of fixed point analysis was postulated. The Banach fixed point theorem is a useful method in metric space theory. Stefan Banach (1892-1945), the developer of functional analysis, developed and presented the theorem in 1922, beginning with the multivalued variant of the BanachCaccioppoli contradiction theory demonstrated by S.B Nadler Jr in 1969. H Scarf proposed the first constructive method of calculating the fixed point of a continuous function in 1973. The fixed points of some relevant single valued mappings are also important because their findings can be applied in architecture, physics, computer science, economics, and telecom. Motivated by this fundamental theory , we initiate some common fixed points theorems on the most happening field of D^{*}metric spaces**.**

2. PRELIMINARIES

2.1 BASIC DEFINITIONS

Definition 2.1.1

Let S be any set. A metric for S is a function ‘f’ with domain $S\times S$ and range contained in $[0,\infty)$ such that

1. $\rho(p,p) = 0 \qquad (p \in S)$

2.$\rho(p,q) > 0 \qquad (p,q \in S, p \neq q)$

3. $\rho(p,q) = \rho(q,p) \quad (p,q \in S)$
4. $\rho(p,q) \leq \rho(p,r) + \rho(r,q)(p,q,r \in S)$ (Triangle Inequality)

Definition 2.1.2

If there exists a$\in$ X such that Sa = Ta = y, then a is called a coincidence point of S and T, while b is called a point of coincidence of T and S. If Sa = Ta = a, then a is called a Common Fixed Point of S and T.

Definition 2.1.3

Let X be a non-empty set. A generalized metric (orD^* metric) on X is a function $D^*: X^3 \to [0, \infty)$ that satisfies the following conditions for each a, b, c, u $\in$ X.

i. $D^*(a, b, c) \geq 0$
ii. $D^*(a, b, c) = 0$ iff a = b = c
iii. $D^*(a, b, c) = D^*(P\{a, b, c\})$, (Symmetry) P is a permutation function.
iv. $D^*(a, b, c) \leq D^*(a, b, u) + D^*(u, c, c)$

The pair (X, D^*) is called generalized metric (or D^* metric) space.

Definition 2.1.4

Let (X, D^*) be a D^* - metric space. D^* is said to be continuous function on X^3 if $\lim_{n\to\infty} D^*(a_n, b_n, c_n) = D^*(a, b, c)$ Whenever a sequence $\{(a_n, b_n, c_n)\}$ in X^3 converges to a point (a, b, c) $\in X^3$, that is

$\lim_{n\to\infty} a_n = a$, $\lim_{n\to\infty} b_n = b$, $\lim_{n\to\infty} c_n = c$.

Definition 2.1.5

Let (X, d_X) and (Y, d_Y) be metric spaces. A function f: X → Y is continuous at

u$\in$ X if for every $\epsilon > 0$ there exists $\delta > 0$ such that $d_X(a,u) < \delta$ implies that

$$d_Y(f(a), f(u)) < \epsilon.$$

Definition 2.1.6

Let X be a nonempty set. A generalized metric (or D^* −metric) on X is a function, $D^*: X^3 \to [0, \infty)$, satisfies the following conditions for each $a, b, c, u \in X$:

1) $D^*(a, b, c) \geq 0$,
2) $D^*(a, b, c) = 0$ if and only if $a = b = c$,

3) $D^*(a,b,c) = D^*(p\{a,b,c\})$, (symmetry) where p is a permutation function,

4) $D^*(a,b,c) \leq D^*(a,b,c) + D^*(u,c,c)$.

The pair (X, D^*) is called a generalized metric (or D^* −metric) space.

Example 2.1.6.1

a) $D^*(a,b,c) = \max\{d(a,b), d(b,c), d(c,a)\}$,

b) $D^*(a,b,c) = d(x,y) + d(y,z) + d(z,x)$.Here, d is the ordinary metric on X.

c) If $X = \mathbb{R}^n$ then we define $D^*(a,b,c) = \left(\|a-b\|^P + \|b-c\|^P + \|c-a\|^P\right)^{\frac{1}{P}}$ for every $p \in \mathbb{R}^+$.

d) If $X = \mathbb{R}$, then we define

$$D^*(a,b,c) = \begin{Bmatrix} 0 & \text{if } a=b=c \\ \max\{a,b,c\} & \text{otherwise} \end{Bmatrix}$$

DEFINITION 2.1.7:

Let (X,D^*) be a D^* −metric space. D^* is said to be a continuous function on X^3 if $\lim_{n\to\infty} D^*(a_n, b_n, c_n) = D^*(a,b,c)$Whenever a sequence $\{(a_n, b_n, z_n)\}$ in X^3 converges to a point $(a,b,c) \in X^3$, that is

$$\lim_{n\to\infty} a_n = a, \lim_{n\to\infty} b_n = b, \lim_{n\to\infty} c_n = c.$$

DEFINITION 2.1.8:

In D^* −metric space (X, D^*), P and R be two mappings into itself.

Then {A,S} is said to be weakly commuting pair if

$$D^*(ASx, SAx, SAx) \leq D^*(Ax, Sx, Sx).$$

For all a ∈ X.Clearly a commuting pair is weakly commuting.

2.2 SOME FIXED AND COMMON FIXED POINT THEOREMS IN METRIC SPACESFOR TWO SELF MAPPINGS

Let (X, d) be a metric space and $T_i (i = 1,2)$ be self mappings of X.The purpose of this chapter is to investigate the fixed and common fixed points of T_i, when the pair$T_i (i = 1,2)$ satisfies a condition of the following type:

$$d(Q_1 a, Q_2 b) \leq \Upsilon\big(d(a, Q_1 a), d(b, Q_2 b) d(a,b)\big) \qquad \forall\, a, b \in X, \qquad (2.1)$$

Where Υ is some real valued function defined on a subset of$R \times R \times R$

Throughout this chapter, (X, d) is a complete metric space, Q is the closure of the set$\{d(a, b): a, b \in X\}$ and $P = Q \times Q \times Q$. A function $\Upsilon: G \to R^{+}$(non-negative reals) is right continuous iff$(u_{n1}, u_{n2}, u_{n3}), (u_1, u_2, u_3) \in G$ and $u_{nk} \downarrow u_k,\ k = 1, 2, 3$ ($\downarrow =$ decreasing),then

$$\Psi(u_{n1}, u_{n2}, u_{n3}) \to \Psi(u_1, u_2, u_3).$$

The function Υ will be called symmetric iff $\Upsilon(u, v, w) = \Upsilon(u, v, w)$ for all$(u, v, w) \in G$.

Further, the mappings $Q_i(i = 1,2)$ satisfy a $I_1, I_2, \Upsilon, k)$ functional inequality iff for each

$i(i = 1,2)$,there is a mapping $I_i: Q_i \times X \to I^{+}$(positive integers) such that if

$n(a) = I_1(T_1, a)$,then

$$d\left(Q_1^{n(x)}x, Q_2^{m(y)}y\right) \le k\Upsilon\left(d\left(x, Q_1^{n(a)}a\right), d\left(b, T_2^{m(b)}y\right), d(a, b)\right), \qquad (2.2)$$

For all $a, b \in X$, where k is a some real constant, and $\Upsilon: G \to R^{+}$ is a symmetric rightcontinuous function. If (2.2) holds for $k = 1$, then (I_1, I_2, Υ) will denote $(I_1, I_2, \Upsilon, 1)$.

Theorem 2.2.1

Considering the mapping $Q_i: X \to X\ (i = 1,2)$, satisfy a (I_1, I_2, Υ, k) functional inequality for some $k < 1$.

If (i) $\Psi(u, v, w) \le \max\{u, v\}, (u, v, u) \in G$, then there exists a $\eta \in X$ such that

$$Q_1^{n(\eta)} = Q_2^{m(\eta)}\eta = \eta. \qquad (2.3)$$

If (ii) $\Upsilon(0, 0, u) \le u$ for each $u \in \mathbb{Q}$, then η is unique satisfying (2.3)

Proof

Suppose $a_0 \in X$ and let $a_1 = Q_1^{n(a_0)}a_0, a_2 = Q_2^{m(a_1)}a_1$, and inductively

$$a_{2n} = Q_2^{m(a_{2n-1})}a_{2n-1}, a_{2n+1} = T_1^{n(a_{2n})}a_{2n}.$$

Then, $\quad d(a_{2n}, a_{2n+1}) \le k\Upsilon\big(d(a_{2n-1}, a_{2n}), (a_{2n}, a_{2n+1}), d(a_{2n-1}, a_{2n})\big)$.

Since $k < 1$, it from by (i) we have

$$d(a_{2n}, a_{2n+1}) \le kd((a_{2n-1}, a_{2n}) \qquad (2.4)$$

In a similar manner

$$d(a_{2n-1}, a_{2n}) \le kd(a_{2n-2}, a_{2n-1}). \qquad (2.5)$$

Hence, $\{d(a_n, a_{n+1})\}$ is non-increasing sequence of reals and it is obvious from (2.4) and (2.5) that

$d(a_n, a_{n+1}) \leq k^n d(a_0, a_1) \to 0 as n \to \infty$.

Therefore, It follows that $\{a_n\}$ is a Cauchy sequence in X.

let $a_n \to \eta$. For proving $Q_1^{n(\eta)} = Q_2^{m(\eta)}\eta = \eta$, choose a subsequence $\{a_{2n(i)+1}\}$ of the sequence $\{a_{2n}\}$ such that $d(a_{2n(i)}, \{\eta\}) \downarrow 0$. Then

$$d(a_{2n(i)+1}, Q_2^{m(\eta)}\eta) \leq k\Upsilon(d(a_{2n(i)}, a_{2n(i)+1}), d\left(\eta, Q_2^{m(\eta)}\eta\right), d(a_{2n(i)}, \eta))$$

Letting $i \to \infty$, we obtain

$$\begin{aligned} d\left(\eta, Q_2^{m(\eta)}\eta\right) &\leq k\Upsilon\left(0, d\left(\eta, Q_2^{m(\eta)}\eta\right), 0\right) \\ &\leq kd\left(\eta, Q_2^{m(\eta)}\eta\right), \end{aligned}$$

i.e) $\left(Q_2^{m(\eta)}\eta\right) = \eta$.

Choosing a subsequence $\{a_{2n(k)+1}\}$ of the sequence $\{a_{2n+1}\}$ such that $d(a_{2n(k)+1}, \eta) \downarrow 0$. Similarly, We obtain

$$Q_1^{n(\eta)}\eta = \eta.$$

Suppose Υ satisfies (ii) and there is a $x \in X$ such that

$$Q_2^{m(x)}u = Q_1^{n(x)} = x.$$

Then, $$\begin{aligned} d(\eta, x) &= d\left(Q_1^{n(\eta)}\eta, T_2^{m(x)}x\right) \\ &\leq k\Upsilon(0,0, d(\eta, x)) \\ &\leq kd(\eta, x). \end{aligned}$$

Hence η is unique element satisfying (2.3).

If $I_i(i = 1,2)$ are the mappings introduced earlier, then we have

Corollary 2.2.2

Let the mapping $Q_i: X \to X(i = 1,2)$ satisfy either of the following conditions

$$d\left(Q_1^{n(a)}x, Q_2^{m(b)}y\right) \leq k \max\left\{d\left(a, Q_1^{n(a)}a\right), d\left(b, Q_2^{m(b)}b\right), d(a, b)\right\} \text{ for some } k < 1, \tag{2.6}$$

$$d\left(Q_1^{n(a)}a, Q_2^{m(b)}b\right) \leq \beta d\left(a, Q_1^{n(a)}a\right), \gamma d\left(b, Q_2^{m(b)}b\right) + \alpha d(a,b), \quad (2.7)$$

For some non negative real β, γ, α satisfying $\beta + \gamma + \alpha < 1$.

Then there exists a unique $\eta \in X$ such that $Q_1^{n(\eta)}\eta = Q_2^{m(\eta)}\eta = \eta$.

Proof

Suppose (2.6) holds, let $\Upsilon(u, v, w) = \max\{u, v, w\}$ in Theorem 2.1. In case of (2.7)

let$k = \beta + \gamma + \alpha$. Then (3.7) implies (3.6) and the desired result follows from previous part.

In the special case when I_1 and I_2 are constant mappings, we have

Theorem 2.2.3

For some positive integers p and q, suppose the mappings $Q_i: X \to X$
$(i = 1,2)$satisfy for all $p, q \in X$,

$$d(Q_1^q a, Q_2^p b) \leq k\Upsilon(d(a, Q_1^q a), d(b, Q_2^p y), d(a,b)) \quad (2.8)$$

Where $k < 1$ and the function $\Upsilon: G \to R^+$is symmetric and right continuous. If Υ satisfies condition (i) and (ii) of Theorem 2.1, then $Q_i(= 1,2)$ have a unique common fixed point $\xi \in X$.

Proof:

Using theorem 2.1, there is a unique $\eta \in X$ such that $T_1^{\eta}\eta = T_2^{\eta}\eta$. It follows from (2.8) that η is unique fixed point of Q_1^n, in fact if $Q_1^n x = x$ for some $x \in X$, then

$$d(x, \eta) = d\left(Q_1^q x, Q_2^p \eta\right)$$

$$\leq k\Upsilon\left(0,0, d(x, \eta)\right)$$

$$\leq kd(x, \eta),$$

That is $\eta = x$.

Since $Q_1^q(Q_1\eta) = Q_1\eta$, we have $Q_1\eta = \eta$.

Similarly $Q_2\xi = \xi$.

Corollary 2.2.4

For some integer p and q, suppose the mappings $Q_i: X \to X$ satisfy the condition

$$d(Q_1^q a, Q_2^p b) \leq k \max\{ d(a, Q_1^q a), d(b, Q_2^p b), d(a,b)\} \qquad (2.9)$$

For $k < 1$ and for all $a, b \in X$.Then$T_i(i = 1,2)$ have a unique common fixed point in X.

Corollary 2.2.5:

For some positive integers m and n, if the mappings $Q_i: X \to X(i = 1,2)$ satisfy the inequality

$$d(Q_1^q a, Q_2^p b) \leq \beta d(a, Q_1^q a) + \gamma d(a, Q_2^p b) + \alpha d(a,b) \qquad (2.10)$$

For some non-negative realsβ, γ, α with $\beta + \gamma + \alpha < 1$, then $Q_i(i = 1,2)$ have a unique common fixed point in X.

3.A Common Fixed Point Theorem in D^*- Metric Spaces

3.1 Introduction

A class of implicit relation: Throughout this section (X,D^*) denotes a D^*-metric space and Φ denotes a family of mappings such that each $\varphi \in \Phi$, $\varphi : (\mathbb{R}^+)^5 \to \mathbb{R}^+$, and φ is continuous and increasing in each coordinate variable. Also ω(v) = φ(v, v, u_1v, u_2v, v) < v for every v ∈R+ where u_1 + u_2= 3.

Example 3.1.1

Let $\varphi:(\mathbb{R}^+)^5 \to \mathbb{R}^+$be defined by

$\varphi(v_1, v_2, v_3, v_4, v_5) = \frac{1}{7}(v_1 + v_2 + v_3 + v_4 + v_5)$.

3.2 MAIN RESULTS

Lemma 3.2.1

If (X, D^*) is a $D^* -$ metric space and If r> 0, then the ball $B_{D^*}(a, r)$ with center a $\in X$ and radius r is open ball.

Proof:

Suppose $c \in B_{D^*}(a, r)$ and hence $D^*(a, c, c) < r$. let $D^*(a, c, c) = \delta$ and $r' = r - \delta$. Let $b \in B_{D^*}(c, r')$.

By triangular inequality we see that,

$$D^*(a, b, b) = D^*(b, b, a) \leq D^*(b, b, c) + D^*(c, a, a) < r' + \delta = r.$$

Therefore $B_{D^*}(c, r') \subseteq B_{D^*}(a, r)$.

Hence the ball $B_{D^*}(a, r)$ is an open ball.

Lemma 3.2.2

$D^*$is said to be a continuous function on X^3 where D^* is metric on X

Lemma 3.2.3

If a sequence $\{a_n\}$ converges to a in a D^* metric space, then a is unique.

Lemma 3.2.4.

If sequence $\{a_n\}$ convergent to a in a D^* $-$metric space, then the sequence $\{a_n\}$ is a Cauchy sequence.

Lemma 3.2.5.

For every $v > 0$, $\omega(v) < v$ if and only if $\lim_{n\to\infty} \omega^n(v) = 0$, where ω^n denotes the composition of ω with itself n times.

Theorem 3.2.6.

Let (X, D^*) be a complete D^* $-$metric space. Let A be a self-mapping of X and let R, Q be continuous self-mappings on X satisfying the following results:

(i) {P, R} and {P, Q} are weakly commuting pairs such that

P(X) ⊂ R(X) ∩ Q(X);

(ii) There exists a $\varphi \in \emptyset$ such that for all $a, b \in X$,

$D^*(Px, Py, Pz) \leq \varphi\big(D^*(Rx, Qy, Qz), D^*(Rx, Py, Py), D^*(Qy, Px, Px), D^*(Qy, Py, Py)\big)$(3.1)

Then P, R and Q have a unique common fixed point in X.

Proof:

Let $Pa_0 \in X$ where a_0 is an arbitrary point. Since P(X) ⊂ R(X), there exists a point $a_1 \in X$ such that Pa_0 = Ra_1.

Also P(X) ⊂ Q(X), we take a point $a_2 \in X$ such that $Pa_1 = Qa_2$. Continuing this way, by applying induction a sequence $\{a_n\}$ in X such that

$$Ra_{2n+1} = Pa_{2n} = b_{2n}, \quad n = 0, 1, 2,\ldots,$$
$$Qa_{2n+2} = Pa_{2n+1} = b_{2n+1}, \quad n = 0, 1, 2,\ldots, \quad (3.2)$$

Now, we setting that

$$d_n = D^*(b_n, b_{n+1}, b_{n+1}), \quad n = 0, 1, 2,\ldots, \quad (3.3)$$

Now we have to prove that $d_{2n} \leq d_{2n-1}$.Since$d_{2n} \geq d_{2n-1}$ for some $n \in \mathbb{N}$. If φ is an increasing function, then

$$d_{2n} = D^*(b_{2n}, b_{2n+1}, b_{2n+1}) = D^*(Px_{2n}, Px_{2n+1}, Px_{2n+1})$$
$$= D^*(Px_{2n+1}, Px_{2n}, Px_{2n})$$
$$\leq \varphi\begin{pmatrix} D^*(Rx_{2n+1}, Qx_{2n}, Qx_{2n}), \quad D^*(Rx_{2n+1}, Px_{2n+1}, Px_{2n+1}), D^*(Rx_{2n+1}, Px_{2n}, Px_{2n}) \\ D^*(Qx_{2n+1}, Px_{2n}, Px_{2n+1}), \quad D^*(Qx_{2n}, Px_{2n}, Px_{2n}) \end{pmatrix}$$

$$= \varphi\begin{pmatrix} D^*(b_{2n}, b_{2n-1}, b_{2n-1}), & D^*(b_{2n}, b_{2n+1}, b_{2n+1}), D^*(b_{2n}, b_{2n}, b_{2n}) \\ D^*(b_{2n-1}, b_{2n+1}, b_{2n+1}), & D^*(b_{2n-1}, b_{2n}, b_{2n}) \end{pmatrix} \quad (3.4)$$

$$D^*(b_{2n-1}, b_{2n+1}, b_{2n+1}) \leq D^*(b_{2n-1}, b_{2n-1}, b_{2n}) + D^*(b_{2n}, b_{2n+1}, b_{2n+1})$$

$$= d_{2n-1} + d_{2n} \quad (3.5)$$

By the above inequality we have

$$d_{2n} \leq \varphi(d_{2n-1}, d_{2n}, 0, d_{2n-1} + d_{2n-1}) \leq \varphi(d_{2n}, d_{2n}, d_{2n}, 2d_{2n}, d_{2n})$$

$$< d_{2n}, \quad (3.6)$$

which arrives at a contradiction. Hence $d_{2n} \leq d_{2n-1}$.

Similarly, we can prove that $d_{2n+1} \leq d_{2n}$ for $n = 0, 1, 2, \ldots$,Consequently, $\{d_n\}$ is a non-increasing sequence of non-negative reals. Here

$$d_1 = D^*(b_1, b_2, b_2)$$

$$= D^*(Pa_1, Pa_2, Pa_2)$$

$$\leq \varphi\begin{pmatrix} D^*(Ra_1, Ra_2, Qa_2), & D^*(Ra_1, Pa_1, Pa_1), D^*(Ra_1, Pa_2, Pa_2) \\ D^*(Qa_{2n}, Pa_1, Pa_1), & D^*(Qa_2, Pa_2, Pa_2) \end{pmatrix}$$

$$= \varphi\begin{pmatrix} D^*(y_0, y_1, y_1), & D^*(y_0, y_1, y_1), D^*(y_0, y_2, y_2) \\ D^*(y_1, y_1, y_1), & D^*(y_1, y_2, y_2) \end{pmatrix} \quad (3.7)$$

$$= \varphi\,(d_0, d_0, d_0 + d_0, 0, d_0)$$

$$\leq \varphi\,(d_0, d_0, 2d_0, d_0, d_0) = \omega(d_0)$$

we have $d_n \leq \omega^n(d_0)$. suppose $d_0 > 0$, lemma 3.5 becomes $\lim_{n\to\infty} d_n = 0$. we clearly have $\lim_{n\to\infty} d_n = 0$, for d_0 =0, since then $d_n = 0$ for each n. Now we show that the sequence $\{Pa_n = b_n\}$ is a Cauchy sequence. Since $\lim_{n\to\infty} d_n = 0$, it is necessary to show that the sequence $\{Pa_{2n} = b_{2n}\}$ is a Cauchy sequence. Assume that $\{Pa_{2n} = b_{2n}\}$ is not a Cauchy sequence. For each even integer $2k$, There is an $\in > 0$ for $k = 0,1,2, \ldots$,then there exist even integers $2n(k)$ and $2m(k)$ with $2k \leq 2n(k) < 2m(k)$ such that

$$D^*\left(Pa_{2q}, Pa_{2q(k)}, Pa_{2p(k)}\right) > \in \quad (3.8)$$

Let, for each even integers $2k, 2p(k)$ be the least integer exceeding $2q(k)$ satisfying above the equation (3.8)

$$D^*\left(Pa_{2q(k)}, Pa_{2q(k)}, Pa_{2p(k)}\right) \leq \in,$$

$$D^*(Pa_{2q(k)}, Pa_{2q(k)}, Pa_{2p(k)}) > \in. \quad (3.9)$$

For each even integer 2k, we get that

$$\in < D^*(Pa_{2q(k)}, Pa_{2q(k)}, Pa_{2p(k)})$$

$$\leq D^*(Pa_{2q(k)}, Pa_{2q(k)}, Pa_{2p(k)-2}) + D^*(Pa_{2p(k)-2}, Pa_{2q(k)-2}, Pa_{2q(k)-1})$$

$$+D^*(Pa_{2p(k)-1}, Pa_{2p(k)-1}, Pa_{2p(k)}) \quad (3.10)$$

$$= D^*(Pa_{2q(k)}, Pa_{2q(k)}, Pa_{2p(k)-2}) + d_{2p(k)-2} + d_{2p(k)-1}$$

Hence by equation and $d_n \to 0$, we have

$$\lim_{k\to\infty} D^*(Pa_{2q(k)}, Pa_{2q(k)}, Pa_{2p(k)}) = \in \quad (3.11)$$

Using triangular inequality, we obtain

$$|D^*(Pa_{2q(k)}, Pa_{2q(k)}, Pa_{2p(k)-1}) - D^*(Pa_{2q(k)}, Pa_{2q(k)}, Pa_{2p(k)})| \leq d_{2p(k)-1},$$

$$|D^*(Pa_{2q(k)+1}, Pa_{2q(k)+1}, Pa_{2p(k)-1}) - D^*(Pa_{2q(k)}, Pa_{2q(k)}, Pa_{2p(k)})$$

$$\leq d_{2p(k)-1} + d_{2q(k)}. \quad (3.12)$$

letting as $k \to \infty$, in equation (3.6), we have

$$D^*(Pa_{2q(k)}, Pa_{2q(k)}, Pa_{2p(k)-1}) \to \in$$

$$D^*(Pa_{2q(k)+1}, Pa_{2q(k)+1}, Pa_{2p(k)-1}) \to \in \quad (3.13)$$

$$D^*(Pa_{2q(k)}, Pa_{2q(k)}, Pa_{2p}) \leq D^*(Pa_{2q(k)}, Pa_{2q(k)}, Pa_{2q(k)+1}) + D^*(Pa_{2q(k)+1}, Pa_{2p(k)}, Pa_{2p(k)})$$

$$\leq d_{2q(k)}$$

$$+\varphi\begin{pmatrix} D^*(Pa_{2q(k)}, Pa_{2p(k)-1}, Pa_{2p(k)-1}, & d_{2q(k)}, & D^*(Pa_{2q(k)}, Pa_{2p(k)}, Pa_{2p(k)})) \\ D^*(Pa_{2p(k)-1}, Pa_{2p(k)+1}, Pa_{2q(k)+1}), & & d_{2p(k)-1} \end{pmatrix}$$

By using (3.13), $\lim_{k\to\infty} d_n = 0$, and continuity and nondecreasing property of φ in each coordinate variable, becomes

$$\in \leq \varphi(\in, 0, \in, \in, 0)$$

$$\leq \varphi(\in, \in, 2\in, \in, \in)$$

$$= \gamma(\in) < \in \quad (3.14)$$

Letting as $k \to \infty$, we arrive at a contradiction. Hence $\{Pa_n = b_n\}$ is a Cauchy sequence and by completeness of X, we can say that it converges to $c \in X$.

That is,

$$\lim_{n\to\infty} Pa_n = \lim_{n\to\infty} b_n = c. \quad (3.15)$$

∵the sequences $\{Ra_{2q+1} = b_{2q+1}\}$ and $\{Qa_{2q} = b_{2q}\}$ are subsequences of

$\{Pa_n = b_n\}$ andthey have the same limit c. Since R and Q are continuous, we have $RQa_{2q} \to Rc$ and $QRa_{q+1} \to Qc$

Let $D^*(RQa_{2q}, QRa_{2q+1}, QRa_{2q+1})$ = $D^*(RPa_{2q-1}, QPa_{2q}, QPa_{2q})$

$$\begin{aligned}\le\ & D^*(RP_{2q-1}, PRa_{2q-1}, PRa_{2q-1}) + \\ & D^*(PRa_{2q-1}, PRa_{2q-1}, PQa_{2q}) + \\ & D^*(PQa_{2q}, PQa_{2q}, QPa_{2q}) \qquad (3.16)\end{aligned}$$

By (ii) and the weak commutativity of {P, R} and {P, Q} we obtain

$$D^*(RQa_{2q}, QRa_{2q+1}, QRa_{2q+1}) \le D^*(Ra_{2q-1}, Pa_{2q-1}, Pa_{2q-1}) + D^*(PRa_{2q}, PQa_{2q}, PQa_{2q})$$
$$+D^*(Pa_{2q}, Pa_{2q}, Qa_{2q})$$

$$\le D^*(Ra_{2q-1}, Pa_{2q-1}, Pa_{2q-1})$$
$$+\ \varphi\begin{pmatrix} D^*(R^2a_{2q-1}, Q^2a_{2q-1}, Q^2a_{2q-1}), & D^*(R^2a_{2q-1}, PRa_{2q-1}, PRa_{2q-1}), D^*(R^2a_{2q-1}, PQa_{2q}, PQa_{2q}) \\ D^*(Q^2a_{2q}, PRa_{2q-1}, PRa_{2q-1}), & D^*(Q^2a_{2q}, PQa_{2q}, PQa_{2q}) \end{pmatrix}$$
$$+D^*(Pa_{2q}, Pa_{2q}, Qa_{2q})$$

$$\le D^*(Ra_{2q-1}, Pa_{2q-1}, Pa_{2q-1}) +$$
$$\varphi\begin{pmatrix} D^*(R^2a_{2q-1}, Q^2b_{2q}, Q^2a_{2q}), D^*(R^2a_{2q-1}, R^2a_{2q-1}, RPa_{2q-1}) + D^*(R^2a_{2q-1}, Q^2a_{2q-1}, Q^2a_{2q-1}) \\ D^*(R^2a_{2q-1}, QPa_{2q}, QPa_{2q}) + D^*(Qa_{2q}, Qa_{2q}, Pa_{2q}), \\ D^*(Q^2a_{2q}, RPa_{2q-1}, RPa_{2q-1}) + D^*(Ra_{2q-1}, Ra_{2q-1}, Pa_{2q-1}), D^*(Q^2a_{2q}, QPa_{2q}, QPa_{2q}) \\ +D^*(Qa_{2q}, Pa_{2q}, Pa_{2q}) \end{pmatrix}$$
$$+D^*(Pa_{2q}, Pa_{2q}, Qa_{2q}). \qquad (3.17)$$

If $D^*(R_c, Q_c, Q_c) > 0$, then letting as $n \to \infty$ we have

$D^*(R_c, Q_c, Q_c)$

$$\leq D^*(c,c,c) + \varphi\begin{pmatrix} D^*(Rc,Qc,Qc), & D^*(Rc,Qc,Qc)+0, D^*(Rc,Qc,Qc)+0 \\ D^*(Qc,Rc,Rc)+0, & D^*(Qc,Qc,Qc)+0 \end{pmatrix} + 0$$

$$\leq \left(D^*(Rc,Qc,Qc)\right) < D^*(Rc,Qc,Qc),$$

we arrive at a contradiction.

Therefore Rc = Qc.

Now, for proving $Pc = Rc$.We define the inequality

$$D^*\left(RPa_{2q+1}, Pc, Pc\right) \leq D^*\left(RPa_{2q+1}, PRa_{2q+1}, PRa_{2q+1}\right) + D^*\left(Pc, Pc, PRa_{2q+1}\right). \quad (3.18)$$

by (ii) and the weak commutativity of {P, R}, we obtain

$$D^*\left(RPa_{2p+1}, Pc, Pc\right) \leq D^*\left(Ra_{2q+1}, Pa_{2q+1}, Pa_{2q+1}\right) + \varphi\begin{pmatrix} D^*\left(Rc,Qc,QRa_{2q+1}\right), & D^*(Rc,Pc,Pc), D^*(Rc,Pc,Pc) \\ D^*(Qc,Qc,Qc), & D^*(Qc,Pc,Pc) \end{pmatrix} \quad (3.19)$$

lettingn $\to \infty$, we get

$$D^*(Rc,Pc,Pc) \leq D^*(c,c,c) + \varphi\begin{pmatrix} D^*(Rc,Qc,Qc), D^*(Rc,Pc,Pc), D^*(Rc,Pc,Pc) \\ D^*(Qc,Pc,Pc), D^*(Qc,Qc,Pc) \end{pmatrix}$$

$$= \varphi\left(0, D^*(Rc,Pc,Pc), D^*(Rc,Rc,Rc), D^*(Rc,Pc,Pc), D^*(Rc,Pc,Pc)\right)$$

$$\leq \delta\left(D^*(Rc,Pc,Pc)\right)$$

$$< D^*(Rc,Pc,Pc) \quad (3.20)$$

Hence by $Rc = Pc$. Therefore$Pc = Rc = Qc$. Hence we follows that

$D^*\left(Pc, Pa_{2q}, Pa_q\right)$

$$\leq \varphi\begin{pmatrix} D^*\left(Rc,Qa_{2q},Qa_{2q}\right), & D^*(Rc,Pc,Pc), D^*\left(Rc,Pa_{2q},Pa_{2q}\right) \\ D^*\left(Qa_{2q},Pc,Pc\right), & D^*\left(Qa_{2q},Pa_{2q},Pa_{2q}\right) \end{pmatrix} \quad (3.21)$$

lettingn $\to \infty$, we have

$$D^*(Pc,c,c) \leq \varphi(D^*(Rc,c,c), 0, D^*(Rc,c,c), D^*(c,Pc,Pc), 0)$$

$$\leq \omega\left(D^*(Pc,c,c)\right) < D^*(Pc,c,c), \quad (3.22)$$

$Pc = c = Rc = Qc$. Therefore c is a common fixed point of $P, R,$ and Q. The unicity of the common fixed point is not hard to verify.

Corollary 3.2.7

Let $P, T, R, Q,$ and U be self-mapping of complete D^* −metric space (X, D^*), and let SR, TH be continuous self-mappings on X satisfying the following conditions:

I) $\{P, RT\}$ and $\{P, QU\}$ are weakly commuting pairs such that

$$P(X) \subset RT(X) \bigcap QU(X);$$

II) There exists a $\varphi \in \emptyset$ such that for all $a, b \in X$,

$$D^*(Pa, Pb, Pc) \leq \varphi\left(\begin{smallmatrix} D^*(TQa,QUb,QUc),D^*(RTa,Pa,Pa),D^*(RTa,Pb,Pb), \\ D^*(QUb,Pa,Pa),D^*(QUb,Pb,PB) \end{smallmatrix}\right) \text{(3.23)}$$

If $RT = TR, QU = UQ, PU = UP$, and $PT = TP$, then $P, R, T, U,$ and Q have a unique common fixed point in X.

Proof:

Using theorem 3.2.6, $P, QU,$ and RT have a unique common fixed point in X.Which implies, there exists $u \in X$, such that $P(u) = QU(u) = RT(u) = u$.

We show that

$$U(u) = u. \text{ By (II)}$$

We obtain

$$D^*(ARa, Aa, Aa) \leq \varphi\left(\begin{smallmatrix} D^*(RTTu,QUu,QUu),D^*(RTTu,PTu,PTu),D^*(RTTu,Pu,Pu), \\ D^*(QUu,PTu,PTu),D^*(QUu,Pu,Pu) \end{smallmatrix}\right) \text{(3.24)}$$

Now, if $Tu \neq u$,then we get

$$\begin{aligned} D^*(Tu, u, u) &\leq \varphi\big(D^*(Tu, u, u), D^*(Tu, Tu, Tu), D^*(Tu, u, u), D^*(u, Tu, Tu), D^*(u, u, u)\big) \\ &\leq \varphi\big(D^*(Tu, u, u), D^*(Tu, u, u), D^*(Tu, u, u), 2D^*(Tu, u, u), D^*(Tu, u, u)\big) \end{aligned}$$

$$< D^*(Tu, u, u), \quad \text{(3.25)}$$

We arrive at a contradiction. Hence $Tu = u$.

Hence $R(u) = RT(u) = u$. Similarly, we have that $Q(u) = U(u) = u$.

Corollary 3.2.8

Let E_i be a sequence self-mapping of complete D^* −metric space $(X, D^*)$for $i \in \mathbb{N}$, and let R, Q be continuous self-mappings on X satisfying the following conditions:

i) $\exists i_0 \in \mathbb{N}$ such that $\{E_{i_0}, R\}$ and $\{P_{i_0}, Q\}$ are weakly commuting pairs such that $E_{i_0}(X) \subset R(X) \cap Q(X)$;

ii) $\exists\varphi \in \emptyset$ and $i, j, k \in \mathbb{N}$ such that for all $a, b \in X$,

$$D^*(E_i a, E_j b, E_k c) \leq \varphi\begin{pmatrix} D^*(Ra,Qb,Qc), D^*(Ra,E_i a,P_i a), D^*(Ra,P_j b,E_j c), \\ D^*(Qb,E_i a,E_i a), D^*(Qb,P_j b,E_j b) \end{pmatrix} \quad (3.26)$$

In this case E_i, S and T have a unicity of common fixed point in X for each $i \in \mathbb{N}$

Proof :

Using the theorem 3.2.6, let $R, Q,$ and E_{i_0} have a unicity of common fixed point in X for some $i = j = k = i_0 \in \mathbb{N}$. Then, there exists a unique $u \in X$ such that

$$R(u) = Q(u) = E_{i_0}(u) = u. \quad (3.27)$$

If there exists $i \in \mathbb{N}$ such that $i \neq i_0$ and $j = i_0, k = i_0$. Then we obtain

$$D^*(E_i u, E_{i_0} u, E_{i_0} u) \leq \varphi\begin{pmatrix} D^*(Ru, Qu, Qu), D^*(Ru, E_i u, E_i u), D^*(Ru, E_{i_0} u, E_{i_0} u), \\ D^*(Qu, P_i u, E_i u), \quad D^*(Qu, E_{i_0} u, E_{i_0} u) \end{pmatrix}. \quad (3.28)$$

Now if $E_i u \neq u$, then we obtain

$$D^*(E_i u, u, u) \leq \varphi\begin{pmatrix} D^*(u,u,u), D^*(u,E_i u,E_i u), D^*(u,u,u), \\ D^*(u,E_i u,E_i u), D^*(E_i u,u,u) \end{pmatrix} \quad (3.29)$$

$$< D^*(E_i u, u, u),$$

We arrive at a contradiction. Hence for every $i \in \mathbb{N}$ it follows that $E_i(u) = u$ for every $i \in \mathbb{N}$.

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