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Optimization Of Fuzzy Integrated Inventory Model Using Triangular And Pentagonal Fuzzy Number

F.S.Josephine , A. Saranya & I. Francina Nishandhi

Department of Mathematics, Holy Cross College (Autonomous), Tiruchirappalli, Tamil Nadu.

Abstract:

Inventory is one of the most expensive and important assets to many companies. In this paper, inventory model for both buyer and vendor are considered together under fuzzy situation and whose parameters are different fuzzy numbers. Aim is to minimize the integrated total cost function. Signed distance method and Graded Mean integration method are used for defuzzification process. A numerical example is given to demonstrate this method.

Keywords: Integrated inventory model, triangular and pentagonal fuzzy number, Lagrangian method.

1. INTRODUCTION

In traditional inventory management systems, the economic-lot-size (ELS) for a vendor and a purchaser are managed independently, that is, the vendor finds their own optimal order quantity. As a result, the ELS of purchaser may not result in an optimal policy for the vendor and vice versa. To overcome this problem, researchers have studied joint economic lot size (JELS) model where the joint total relevant cost (JTRC) for the purchaser as well as the vendor has been optimized. Goyal [1] first introduced an integrated inventory policy for a single supplier and a single customer and derived the minimum joint variable cost for the supplier and the customer. Banerjee [2] introduced the JELS model for a single vendor and a single customer and obtained the minimum joint total relevant cost for both buyer and vendor at the same time with the assumption that the vendor makes the production set up every time the buyer places an order and supplies on a lot for lot basis. An integrated inventory model that allows the two trading partier to form a strategic alliance for profit sharing may prove helpful in treating down the traditional barrier.

Various types of uncertainties and imprecision is inherent in real problems. They are classically modeled using the approaches from the probability theory. However, there are uncertainties that cannot be appropriately treated by usual probabilistic models. The question arise how to define inventory optimization tasks in such environment and how to interpret optimal solutions. Therefore it becomes move convenient to deal such problems with fuzzy set theory rather than probability theory. Fuzzy concepts have introduced in EOQ models. Park developed a fuzzy EOQ model using extension principle. In 1999, Chang presented a membership function of the fuzzy total cost of production inventory model and use extension principle and centroid method to obtain an estimate of the total cost and to obtain an estimate of the total cost and to obtain the economic production quantity.

2. PRELIMINARIES

2.1 Fuzzy set

Let A be a classical set, $\mu_{\bar{A}}(x)$ be a function from A to $[0,1]$. A **fuzzy set** $\bar{A}$ with the membership function $\mu_{\bar{A}}(x)$ is defined by

$$\bar{A} = \{x, \mu_{\bar{A}}(x)); x \epsilon A, \mu_{\bar{A}}(x) \epsilon [0,1] \}.$$

2.2 Fuzzy Number

A fuzzy subset of real number with membership function $\mu_{\widetilde{A}} : R \rightarrow [0,1]$ is called a **fuzzy number** if

1. $\widetilde{A}$ is normal, that is there exists an element X_0 such that $\mu_{\widetilde{A}}(x_0) = 1$
2. $\widetilde{A}$ is convex that is $\mu_{\widetilde{A}}(\lambda x_1 + (1-\lambda) x_2) \geq \mu_{\widetilde{A}}(x_1) \wedge \mu_{\widetilde{A}}(x_2)\ \forall\ x_1, x_2 \in R\ \&\ \lambda \in [0,1]$
3. $\mu_{\widetilde{A}}$ is upper semi continuous;
4. Supp $(\widetilde{A})$ is bounded, here Supp $(\widetilde{A})$ = supp $\{X \in R : \mu_{\widetilde{A}}(x) > 0\}$.

2.3 Triangular Fuzzy number

A fuzzy number $\widetilde{A} = (a_1, a_2, a_3)$ with $a_1 < a_2 < a_3$ is triangular if its membership function is defined as

$$\mu_{\widetilde{A}}(x) = \begin{cases} \frac{x-a_1}{a_2-a_1}, & \text{when } a_1 \leq x \leq a_2 \\ \frac{a_3-x}{a_3-a_2}, & \text{when } a_2 \leq x \leq a_3 \\ 0 & \text{otherwise} \end{cases}$$

2.4 Pentagonal Fuzzy Number

A pentagonal fuzzy number $\widetilde{A} = (a,b,c,d,e)$ is represented with membership function $\mu_{\widetilde{A}}(x)$ as

$$\mu_{\widetilde{A}}(x) = \begin{cases} \frac{x-a}{b-a}, & \text{when } a \leq x \leq b \\ \frac{x-b}{c-b}, & \text{when } b \leq x \leq c \\ 1, & \text{when } x=c \\ \frac{d-x}{d-c}, & \text{when } c \leq x \leq d \\ \frac{e-x}{e-d}, & \text{when } d \leq x \leq e \\ 0 & \text{otherwise} \end{cases}$$

2.5 Fuzzy arithmetical operations in Triangular Fuzzy number

Suppose $\widetilde{A} = (a_1, a_2, a_3)$ and $\widetilde{B} = (b_1, b_2, b_3)$ are triangular fuzzy numbers then the arithmetical operations are defined as

i. **Addition**

$\widetilde{A} + \widetilde{B} = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$

ii. **Subtraction**

$\widetilde{A} - \widetilde{B} = (a_1 - b_3, a_2 - b_2, a_3 - b_1)$

iii. **Multiplication**

$\widetilde{A} \times \widetilde{B} = (a_1 b_1, a_2 b_2, a_3 b_3)$

iv. **Division**

$$\tilde{A} \oslash \tilde{B} = \left(\frac{a_1}{b_3}, \frac{a_2}{b_2}, \frac{a_3}{b_1}\right)$$

v. Scalar multiplication

$$\alpha \times \tilde{A} = \begin{cases} (\alpha a_1, \ \alpha a_2, \ \alpha a_3), \alpha \geq 0 \\ (\alpha a_3, \alpha a_2, \alpha a_3), \quad \alpha < 0 \end{cases}$$

2.6 Defuzzification:

Let $\tilde{A}$ be a fuzzy set defined on R. then the signed distance of $\tilde{A}$ is defined as ,

$d_F(\tilde{A}, 0) = \frac{a_1+2a_2+a_3}{4}$ for defuzzifying triangular fuzzy number $\tilde{A}$= (a_1 , a_2 , a_3).

Graded mean integration representation for defuzzifying the Pentagonal fuzzy number $\tilde{A}$= (a_1 , a_2 , a_3, a_4 ,a_5) is defined as $d_F(\tilde{A}) = \frac{a_1+3a_2+4a_3+3a_4+a_5}{12}$

3. CRISP INTEGRATED INVENTORY MODEL

This section gives the classical integrated inventory model for both buyer and vendor**.**

3.1 Assumption and Notations:

Following assumption and notation are considered

3.1.1 Assumptions:

1) The demand rate and production rate production rate are deterministic.
2) Manufacturing set up cost , ordering cost , unit inventory holding cost for the vendor and the buyer's ,are known
   Single vendor and single buyer are considered
3) There is a single product
4) Shortage are allowed
5) The vendor makes the production set up every time the buyer places an order and supplies on a lot for lot basis.

3.1.2 Notations

d : demand

P_r : Rate of production

C_v : The production cost per unit

P_c : The purchase cost paid by the purchaser per unit

A: The purchaser's ordering cost per order

S_v : The vendor's set up cost per set up

r : Annual inventory carrying cost per dollar

q: The order quantity

3.2 THE MODEL FORMULATION

Based on the above notation and assumption, the total expected annual cost for the purchaser is given by TEP_c= Ordering cost + Holding cost

Since A is the ordering cost per order, the expected ordering cost per year is given by $\left(\frac{d}{q}\right)$A. The expected holding cost per year is $rp_c\left(\frac{q}{2}\right)$. Hence the total expected annual cost for the purchaser is given by $TEP_c\ (q)=\left(\frac{d}{q}\right)A+rp_c\left(\frac{q}{2}\right)$.

For the vendor's inventory model, its expected annual cost can be represented by

TEV_c = Set-up cost + Holding cost.

Since the vendor's setup cost per up and the production quantity for the vendor in a lot will be mq, its expected set up cost per year is given by$\left(\frac{d}{mq}\right)S_v$. Here we consider m=1, where m is an integer

Hence its expected set up cost per year in given by$\left(\frac{d}{q}\right)S_v$.

The vendor produces the item in the quantity of mq, and the Purchaser would receive it m lots, with which each having a quantity of q.

Average inventory cost for vendor is as follows,

$$I_v=\frac{\left\{\left[mq\left(\frac{q}{P_r}+(m-1)\frac{q}{d}\right)-\frac{m^2q^2}{2P_r}\right]-\left[\frac{q}{d}(1+2+\cdots+(m-1)q)\right]\right\}}{\frac{mq}{d}}$$

$$=\frac{q}{2}\left(m\left(1-\frac{d}{P_r}\right)-1+\frac{2d}{P_r}\right)$$

Since m=1 We have the vendor's holding cost per year is $rV_c\left(\frac{q}{2}\right)\left(\frac{d}{P_r}\right)$

Hence the total expected annual cost for the vendor is $TEV_c=\left(\frac{d}{q}\right)S_v+rV_c\left(\frac{q}{2}\right)\left(\frac{d}{P_r}\right)$
Hence the joint relevant cost is given by $F(q)=\left(\frac{d}{q}\right)(S_v+A)+\frac{qr}{2}\left(\frac{d}{P_r}V_c+P_c\right)$

The objective is to find the optimal order quantity which minimizes the joint relevant total cost.

On considering the derivative with respect to q and equated to zero we get,

$$q=\sqrt{\frac{2d(S_v+A)}{r\left(\frac{d}{P_r}V_c+P_c\right)}}$$

4. FUZZY INTEGRATED INVENTORY MODEL WITH FUZZY ORDER QUANTITY

4.1 Fuzzy Integrated Inventory Model with Fuzzy Order Quantity using Triangular Fuzzy Number

In this section, we consider the integrated inventory model with all parameters as fuzzy and they are represented by non-negative triangular fuzzy numbers as follows.

$\tilde{d} = (d_1, d_2, d_3)$ $\tilde{S}_v = (S_{v_1}, S_{v_2}, S_{v_3})$ $\tilde{V}_c = (V_{c_1}, V_{c_2}, V_{c_3})$ $\tilde{P}_r = (P_{r_1}, P_{r_2}, P_{r_3})$

$\tilde{P}_c = (P_{c_1}, P_{c_2}, P_{c_3})$ $\tilde{A} = (A_1, A_2, A_2)$ $\tilde{r} = (r_1, r_2, r_3)$

Also the order quantity is represented by the triangular fuzzy number $\tilde{q} = (q_1, q_2, q_3)$ with

$0 \leq q_1 \leq q_2 \leq q_3$.

Fuzzy total cost for purchaser = $[(\tilde{d} \oslash \tilde{q}) \otimes \tilde{A}] \oplus \left[\tilde{r} \otimes \tilde{P}_c \otimes \frac{\tilde{q}}{2}\right]$

Fuzzy total cost for vendor = $[(\tilde{d} \oslash \tilde{q}) \otimes \tilde{S}_v] \oplus \left[\frac{\tilde{q}}{2} \otimes \tilde{r} \otimes \tilde{V}_c(\tilde{d} \oslash \tilde{P}_r)\right]$

The fuzzy total relevant cost of this model is

$$\tilde{F}(\tilde{q}) == [(\tilde{d} \oslash \tilde{q}) \otimes (\tilde{S}_v \oplus \tilde{A})] \oplus \left[\frac{\tilde{q}}{2} \otimes \tilde{r} \otimes (\tilde{d} \oslash \tilde{P}_r) \otimes \tilde{V}_c \oplus \tilde{P}_c\right]$$

Which is reduced to triangular fuzzy number $\tilde{F}(\tilde{q}) = (F_1, F_2, F_3)$ Where

$$F_1 = \frac{d_1 (S_{v_1} + A_1)}{q_3} + \frac{q_1 r_1}{2}\left(\frac{d_1 V_{c_1}}{P_{r_3}} + P_{c_1}\right) \qquad F_2 = \frac{d_2 (S_{v_2} + A_2)}{q_2} + \frac{q_2 r_2}{2}\left(\frac{d_2 V_{c_2}}{P_{r_2}} + P_{c_2}\right)$$

$$F_3 = \frac{d_3 (S_{v_3} + A_3)}{q_1} + \frac{q_3 r_3}{2}\left(\frac{d_3 V_{c_3}}{P_{r_1}} + P_{c_3}\right)$$

After defuzzifying $\tilde{F}(\tilde{q})$ by signed distance method we get $P[\tilde{F}(\tilde{q})] = \frac{1}{4}[F_1 + 2F_2 + F_3]$

$$= \frac{1}{4}\left\{\left[\frac{d_1 (S_{v_1} + A_1)}{q_3} + \frac{q_1 r_1}{2}\left(\frac{d_1 V_{c_1}}{P_{r_3}} + P_{c_1}\right) + 2\left[\frac{d_2 (S_{v_2} + A_2)}{q_2} + \frac{q_2 r_2}{2}\left(\frac{d_2 V_{c_2}}{P_{r_2}} + P_{c_2}\right)\right] + \left[\frac{d_3 (S_{v_3} + A_3)}{q_1} + \frac{q_3 r_3}{2}\left(\frac{d_3 V_{c_3}}{P_{r_1}} + P_{c_3}\right)\right]\right]\right\}$$

Now differentiating $P[\tilde{F}(\tilde{q})]$ partially with respect to q_1 and equate to zero we get,

$$\frac{\partial(P[\tilde{F}(\tilde{q})])}{\partial q_1} = \frac{1}{4}\left[\frac{-d_3 (S_{v_3} + A_3)}{q_1^2} + \frac{r_1}{2}\left(\frac{d_1 V_{c_1}}{P_{r_3}} + P_{c_1}\right)\right]$$

$\frac{\partial(P[\tilde{F}(\tilde{q})])}{\partial q_1} = 0$ We get, $q_1 = \sqrt{\frac{2d_3 (S_{v_3} + A_3)}{r_1\left(\frac{d_1 V_{c_1}}{P_{r_3}} + P_{c_1}\right)}}$

Similarly we obtain, $q_2 = \sqrt{\frac{2\, d_2 (S_{v_2} + A_2)}{r_2\left(\frac{d_2 V_{c_2}}{P_{r_2}} + P_{c_2}\right)}}$ and $q_3 = \sqrt{\frac{2d_1 (S_{v_1} + A_1)}{r_3\left(\frac{d_3 V_{c_3}}{P_{r_1}} + P_{c_3}\right)}}$

We see that $q_1>q_2>q_3$ hence it does not satisfy the constraint $0<q_1 \leq q_2 \leq q_3$.

Hence we adopt the Lagrangian method to find the solution of q_1, q_2 and q_3

So we convert the inequality constraint $q_2 - q_1 \geq 0$ in to the equality constraint $q_2 - q_1 = 0$ and then minimize $P[\tilde{F}(\tilde{q})]$ subject to $q_2 - q_1 = 0$.we have the lagrangian function as

$L(q_1, q_2, q_3, \lambda) = P[\tilde{F}(\tilde{q})] - \lambda(q_2 - q_1)$ Where, λ are the Lagrangian Multipliers

Differentiate partially with respect to q and equate to zero we get,

$$\frac{\partial L}{\partial q_1} = \frac{1}{4}\left[\frac{-d_3\left(S_{v_3} + A_3\right)}{q_1^2} + \frac{r_1}{2}\left(\frac{d_1 V_{c_1}}{P_{r_3}} + P_{c_1}\right)\right] + \lambda$$

$$\frac{\partial L}{\partial q_2} = \frac{2}{4}\left[\frac{-d_2\left(S_{v_2} + A_2\right)}{q_2^2} + \frac{r_2}{2}\left(\frac{d_2 V_{c_2}}{P_{r_2}} + P_{c_2}\right)\right] - \lambda$$

$$\frac{\partial L}{\partial q_3} = \frac{1}{4}\left[\frac{-d_1\left(S_{v_1} + A_1\right)}{q_3^2} + \frac{r_3}{2}\left(\frac{d_3 V_{c_3}}{P_{r_1}} + P_{c_3}\right)\right]$$

$$\frac{\partial L}{\partial \lambda} = q_2 - q_1$$

Equating to zero we get,

$$q_1 = q_2 = \sqrt{\frac{2\left[d_3\left(S_{v_3} + A_3\right) + 2d_2\left(S_{v_2} + A_2\right)\right]}{r_1\left(\frac{d_1 V_{c_1}}{P_{r_3}} + P_{c_1}\right) + 2r_2\left(\frac{d_2 V_{c_2}}{P_{r_2}} + P_{c_2}\right)}} \qquad q_3 = \sqrt{\frac{2d_1\left(S_{v_1} + A_1\right)}{r_3\left(\frac{d_3 V_{c_3}}{P_{r_1}} + P_{c_3}\right)}}$$

$$L(q_1, q_2, q_3, \lambda) = P[\tilde{F}(\tilde{q})] - \lambda(q_2 - q_1) - \lambda(q_3 - q_2)$$

$$\frac{\partial L}{\partial q_1} = \frac{1}{4}\left[\frac{-d_3\left(S_{v_3} + A_3\right)}{q_1^2} + \frac{r_1}{2}\left(\frac{d_1 V_{c_1}}{P_{r_3}} + P_{c_1}\right)\right] + \lambda$$

$$\frac{\partial L}{\partial q_2} = \frac{2}{4}\left[\frac{-d_2\left(S_{v_2} + A_2\right)}{q_2^2} + \frac{r_2}{2}\left(\frac{d_2 V_{c_2}}{P_{r_2}} + P_{c_2}\right)\right] - \lambda + \lambda$$

$$\frac{\partial L}{\partial q_3} = \frac{1}{4}\left[\frac{-d_1\left(S_{v_1} + A_1\right)}{q_3^2} + \frac{r_3}{2}\left(\frac{d_3 V_{c_3}}{P_{r_1}} + P_{c_3}\right)\right] - \lambda$$

Equating to zero we get,

$$q_1 = q_2 = q_3 = q^* = \sqrt{\frac{2\left[d_3\left(S_{v_3} + A_3\right) + 2d_2\left(S_{v_2} + A_2\right) + d_1\left(S_{v_1} + A_1\right)\right]}{r_1\left(\frac{d_1 V_{c_1}}{P_{r_3}} + P_{c_1}\right) + 2r_2\left(\frac{d_2 V_{c_2}}{P_{r_2}} + P_{c_2}\right) + r_3\left(\frac{d_3 V_{c_3}}{P_{r_1}} + P_{c_3}\right)}}$$

Hence we can get q^* is an optimal solution to problem.

4.2 Fuzzy Integrated Inventory Model with Fuzzy Order Quantity using Pentagonal Fuzzy

Number

In this section, we consider the integrated inventory model with all parameters as fuzzy and they are represented by non-negative Pentagonal fuzzy numbers as follows.

$\tilde{d}=(d_1, d_2, d_3, d_4, d_5)$ $\tilde{S}_v=(S_{v_1}, S_{v_2}, S_{v_3}, S_{v_4}, S_{v_5})$ $\tilde{V_c}=(V_{c_1}, V_{c_2}, V_{c_3}, V_{c_4}, V_{c_5},)$

$\tilde{P}_r=(P_{r_1}, P_{r_2}, P_{r_3}, P_{r_4}, P_{r_5})$ $\tilde{P}_c=(P_{c_1}, P_{c_2}, P_{c_3}, P_{c_4}, P_{c_5})$ $\tilde{A}=(A_1, A_2, A_2, A_4, A_5)$ $\tilde{r}=(r_1, r_2, r_3, r_4, r_5)$

Also the order quantity is represented by the pentagonal fuzzy number $\tilde{q}=(q_1, q_2, q_3, q_4, q_5)$ with

$0\leq q_1 \leq q_2 \leq q_3 \leq q_4 \leq q_5$.

Fuzzy total cost for purchaser = $\left[(\tilde{d} \oslash \tilde{q}) \otimes \tilde{A}\right] \oplus \left[\tilde{r} \otimes \tilde{P}_c \otimes \frac{\tilde{q}}{2}\right]$

Fuzzy total cost for vendor = $\left[(\tilde{d} \oslash \tilde{q}) \otimes \tilde{S}_v\right] \oplus \left[\frac{\tilde{q}}{2} \otimes \tilde{r} \otimes \tilde{V}_c(\tilde{d} \oslash \tilde{P}_r)\right]$

The fuzzy total relevant cost of this model is

$\tilde{F}(\tilde{q})==\left[(\tilde{d} \oslash \tilde{q}) \otimes (\tilde{S}_v \oplus \tilde{A})\right] \oplus \left[\frac{\tilde{q}}{2} \otimes \tilde{r} \otimes (\tilde{d} \oslash \tilde{P}_r) \otimes \tilde{V}_c \oplus \tilde{P}_c\right]$

Which is reduced to pentagonal fuzzy number $\tilde{F}(\tilde{q}) = (F_1, F_2, F_3, F_4, F_5)$ Where

$$F_1=\left[\frac{d_1\left(S_{v_1}+A_1\right)}{q_5}+\frac{q_1 r_1}{2}\left(\frac{d_1 V_{c_1}}{P_{r_5}}+P_{c_1}\right)\right] \qquad F_2=\left[\frac{d_2\left(S_{v_2}+A_2\right)}{q_4}+\frac{q_2 r_2}{2}\left(\frac{d_2 V_{c_2}}{P_{r_4}}+P_{c_2}\right)\right]$$

$$F_3=\left[\frac{d_3\left(S_{v_3}+A_3\right)}{q_3}+\frac{q_3 r_3}{2}\left(\frac{d_3 V_{c_3}}{P_{r_3}}+P_{c_3}\right)\right] \qquad F_4=\left[\frac{d_4\left(S_{v_4}+A_4\right)}{q_2}+\frac{q_4 r_4}{2}\left(\frac{d_4 V_{c_4}}{P_{r_2}}+P_{c_4}\right)\right]$$

$$F_5=\left[\frac{d_5\left(S_{v_5}+A_5\right)}{q_1}+\frac{q_5 r_5}{2}\left(\frac{d_5 V_{c_5}}{P_{r_1}}+P_{c_5}\right)\right]$$

$$P[\tilde{F}(\tilde{q})]=\frac{1}{12}\left(F_1+3F_2+4F_3+3F_4+F_5\right)$$

$$=\frac{1}{12}\left[\frac{d_1\left(S_{v_1}+A_1\right)}{q_5}+\frac{q_1 r_1}{2}\left(\frac{d_1 V_{c_1}}{P_{r_5}}+P_{c_1}\right)\right]+\frac{3}{12}\left[\frac{d_2\left(S_{v_2}+A_2\right)}{q_4}+\frac{q_2 r_2}{2}\left(\frac{d_2 V_{c_2}}{P_{r_4}}+P_{c_2}\right)\right]$$

$$+\frac{4}{12}\left[\frac{d_3\left(S_{v_3}+A_3\right)}{q_3}+\frac{q_3 r_3}{2}\left(\frac{d_3 V_{c_3}}{P_{r_3}}+P_{c_3}\right)\right]+\frac{3}{12}\left[\frac{d_4\left(S_{v_4}+A_4\right)}{q_2}+\frac{q_4 r_4}{2}\left(\frac{d_4 V_{c_4}}{P_{r_2}}+P_{c_4}\right)\right]$$

$$+\frac{1}{12}\left[\frac{d_5\left(S_{v_5}+A_5\right)}{q_1}+\frac{q_5 r_5}{2}\left(\frac{d_5 V_{c_5}}{P_{r_1}}+P_{c_5}\right)\right]$$

Differentiate with respect to q_1 and equating to zero we get,

$$\frac{\partial\left(P[\tilde{F}(\tilde{q})]\right)}{\partial q_1}=\frac{1}{12}\left[\frac{-d_5\left(S_{v_5}+A_5\right)}{q_1^2}+\frac{r_1}{2}\left(\frac{d_1 V_{c_1}}{P_{r_5}}+P_{c_1}\right)\right]$$

$$\frac{1}{12}\left[\frac{-d_5\left(S_{v_5}+A_5\right)}{q_1^2}+\frac{r_1}{2}\left(\frac{d_1 V_{c_1}}{P_{r_5}}+P_{c_1}\right)\right]=0 \quad \text{we get} \quad q_1=\sqrt{\frac{2d_5\left(S_{v_5}+A_5\right)}{r_1\left(\frac{d_1 V_{c_1}}{P_{r_5}}+P_{c_1}\right)}}$$

Similarly, we get

$$q_2 = \sqrt{\frac{2d_4\left(S_{v_4}+A_4\right)}{r_2\left(\frac{d_2 V_{c_2}}{P_{r_4}}+P_{c_2}\right)}} \quad q_3 = \sqrt{\frac{2d_3\left(S_{v_3}+A_3\right)}{r_3\left(\frac{d_3 V_{c_3}}{P_{r_3}}+P_{c_3}\right)}} \quad q_4 = \sqrt{\frac{2d_2\left(S_{v_2}+A_2\right)}{r_4\left(\frac{d_4 V_{c_4}}{P_{r_2}}+P_{c_4}\right)}} \quad q_5 = \sqrt{\frac{2d_1\left(S_{v_1}+A_1\right)}{r_5\left(\frac{d_5 V_{c_5}}{P_{r_1}}+P_{c_5}\right)}}$$

We see that $q_1>q_2>q_3>q_4>q_5$ hence it does not satisfy the constraint $0<q_1 \leq q_2 \leq q_3 \leq q_4 \leq q_5$. Hence we adopt the Lagrangian method to find the solution of q_1, q_2, q_3, q_4 and q_5.

So we convert the inequality constraint $q_2 - q_1 \geq 0$ in to the equality constraint $q_2 - q_1 = 0$ and then minimize $P[\tilde{F}(\tilde{q})]$ subject to $q_2 - q_1 = 0$.we have the lagrangian function as

$L(q_1, q_2, q_3, \lambda) = P[\tilde{F}(\tilde{q}) -]\lambda(q_2 - q_1)$ Where λ is the Lagrangian multipliers

Differentiate$\tilde{F}(\tilde{q})$ partially with respect to q and equate to zero we get,

$$\frac{\partial L}{\partial q_1} = \frac{1}{12}\left[\frac{-d_5\left(S_{v_5}+A_3\right)}{q_1^2} + \frac{r_1}{2}\left(\frac{d_1 V_{c_1}}{P_{r_5}} + P_{c_1}\right)\right] + \lambda$$

$$\frac{\partial L}{\partial q_2} = \frac{3}{12}\left[\frac{-d_4\left(S_{v_4}+A_4\right)}{q_2^2} + \frac{r_2}{2}\left(\frac{d_2 V_{c_2}}{P_{r_4}} + P_{c_2}\right)\right] - \lambda$$

$$\frac{\partial L}{\partial q_3} = \frac{1}{4}\left[\frac{-d_3\left(S_{v_3}+A_3\right)}{q_3^2} + \frac{r_3}{2}\left(\frac{d_3 V_{c_3}}{P_{r_3}} + P_{c_3}\right)\right] \quad \frac{\partial L}{\partial q_4} = \frac{1}{4}\left[\frac{-d_2\left(S_{v_2}+A_2\right)}{q_4^2} + \frac{r_4}{2}\left(\frac{d_4 V_{c_4}}{P_{r_2}} + P_{c_4}\right)\right]$$

$$\frac{\partial L}{\partial q_5} = \frac{1}{4}\left[\frac{-d_1\left(S_{v_1}+A_1\right)}{q_5^2} + \frac{r_5}{2}\left(\frac{d_5 V_{c_5}}{P_{r_1}} + P_{c_5}\right)\right]$$

$$\frac{\partial L}{\partial \lambda} = q_2 - q_1$$

Equating to zero we get,

$$q_1 = q_2 = \sqrt{\frac{2\left(d_5\left(S_{v_5}+A_5\right) + 3d_4\left(S_{v_4}+A_4\right)\right)}{r_1\left(\frac{d_1 V_{c_1}}{P_{r_5}} + P_{c_1}\right) + 3r_2\left(\frac{d_2 V_{c_2}}{P_{r_4}} + P_{c_2}\right)}}$$

$$q_3 = \sqrt{\frac{2d_3\left(S_{v_3}+A_3\right)}{\left(r_3\left(\frac{d_3 V_{c_3}}{P_{r_3}} + P_{c_3}\right)\right)}} \quad q_4 = \sqrt{\frac{2d_2\left(S_{v_2}+A_2\right)}{r_4\left(\frac{d_4 V_{c_4}}{P_{r_2}} + P_{c_4}\right)}} \quad q_5 = \sqrt{\frac{2d_1\left(S_{v_1}+A_1\right)}{r_5\left(\frac{d_5 V_{c_5}}{P_{r_1}} + P_{c_5}\right)}}$$

Now the Lagrangian function with multipliers λ_1 and λ_2.

$$L(q_1, q_2, q_3, q_4, q_5, \lambda) = P[\tilde{F}(\tilde{q})] - \lambda_1(q_2 - q_1) - \lambda_2(q_3 - q_2)$$

We can obtain a solution from differentiating $L(q_1, q_2, q_3, q_4, q_5, \lambda_1, \lambda_2)$ with respect to q and equating to zero we get,

Now the Lagrangian function with multipliers λ_1 , , λ_2 and λ_3

$$\frac{\partial L}{\partial q_1} = \frac{1}{12}\left[\frac{-d_5\left(S_{v_5} + A_3\right)}{q_1^2} + \frac{r_1}{2}\left(\frac{d_1 V_{c_1}}{P_{r_5}} + P_{c_1}\right)\right] + \lambda_1$$

$$\frac{\partial L}{\partial q_2} = \frac{3}{12}\left[\frac{-d_4\left(S_{v_4} + A_4\right)}{q_2^2} + \frac{r_2}{2}\left(\frac{d_2 V_{c_2}}{P_{r_4}} + P_{c_2}\right)\right] - \lambda_1 + \lambda_2$$

$$\frac{\partial L}{\partial q_3} = \frac{1}{4}\left[\frac{-d_3\left(S_{v_3} + A_3\right)}{q_3^2} + \frac{r_3}{2}\left(\frac{d_3 V_{c_3}}{P_{r_3}} + P_{c_3}\right)\right] - \lambda_2$$

$$\frac{\partial L}{\partial q_4} = \frac{1}{4}\left[\frac{-d_2\left(S_{v_2} + A_2\right)}{q_4^2} + \frac{r_4}{2}\left(\frac{d_4 V_{c_4}}{P_{r_2}} + P_{c_4}\right)\right] \qquad \frac{\partial L}{\partial q_5} = \frac{1}{4}\left[\frac{-d_1\left(S_{v_1} + A_1\right)}{q_5^2} + \frac{r_5}{2}\left(\frac{d_5 V_{c_5}}{P_{r_1}} + P_{c_5}\right)\right]$$

$$\frac{\partial L}{\partial \lambda} = q_2 - q_1; \ \frac{\partial L}{\partial \lambda_2} = q_3 - q_2$$

$$q_1 = q_2 = q_3 = \sqrt{\frac{2\left(d_5\left(S_{v_5} + A_5\right) + 3d_4\left(S_{v_4} + A_4\right) + 4d_3\left(S_{v_3} + A_3\right)\right)}{r_1\left(\frac{d_1 V_{c_1}}{P_{r_5}} + P_{c_1}\right) + 3r_2\left(\frac{d_1 V_{c_1}}{P_{r_4}} + P_{c_2}\right) + 4r_3\left(\frac{d_3 V_{c_3}}{P_{r_3}} + P_{c_3}\right)}}$$

$$q_4 = \sqrt{\frac{2d_2\left(S_{v_2} + A_2\right)}{r_4\left(\frac{d_4 V_{c_4}}{P_{r_2}} + P_{c_4}\right)}} \qquad q_5 = \sqrt{\frac{2d_1\left(S_{v_1} + A_1\right)}{r_5\left(\frac{d_5 V_{c_5}}{P_{r_1}} + P_{c_5}\right)}}$$

$$L\left(q_1, q_2, q_3, q_4, q_5, \lambda\right) = P[\tilde{F}(\tilde{q})] - \lambda_1(q_2 - q_1) - \lambda_2(q_3 - q_2) - \lambda_3(q_4 - q_3)$$

We can obtain a solution from differentiating $L\left(q_1, q_2, q_3, q_4, q_5, \lambda_1, \lambda_2, \lambda_3\right)$ with respect to q and equating to zero we get,

$$\frac{\partial L}{\partial q_1} = \frac{1}{12}\left[\frac{-d_5\left(S_{v_5} + A_3\right)}{q_1^2} + \frac{r_1}{2}\left(\frac{d_1 V_{c_1}}{P_{r_5}} + P_{c_1}\right)\right] + \lambda_1$$

$$\frac{\partial L}{\partial q_2} = \frac{3}{12}\left[\frac{-d_4\left(S_{v_4} + A_4\right)}{q_2^2} + \frac{r_2}{2}\left(\frac{d_2 V_{c_2}}{P_{r_4}} + P_{c_2}\right)\right] - \lambda_1 + \lambda_2$$

$$\frac{\partial L}{\partial q_3} = \frac{1}{4}\left[\frac{-d_3\left(S_{v_3} + A_3\right)}{q_3^2} + \frac{r_3}{2}\left(\frac{d_3 V_{c_3}}{P_{r_3}} + P_{c_3}\right)\right] - \lambda_2 + \lambda_3$$

$$\frac{\partial L}{\partial q_4} = \frac{1}{4}\left[\frac{-d_2\left(S_{v_2} + A_2\right)}{q_4^2} + \frac{r_4}{2}\left(\frac{d_4 V_{c_4}}{P_{r_2}} + P_{c_4}\right)\right] - \lambda_3$$

$$\frac{\partial L}{\partial q_5} = \frac{1}{4}\left[\frac{-d_1\left(S_{v_1} + A_1\right)}{q_5^2} + \frac{r_5}{2}\left(\frac{d_5 V_{c_5}}{P_{r_1}} + P_{c_5}\right)\right]$$

$$\frac{\partial L}{\partial \lambda} = q_2 - q_1; \ \frac{\partial L}{\partial \lambda_2} = q_3 - q_2; \ \frac{\partial L}{\partial \lambda_3} = q_4 - q_3$$

$$q_1 = q_2 = q_3 = q_4 = \sqrt{\frac{2\left(d_5\left(S_{v_5}+A_5\right)+3d_4\left(S_{v_4}+A_4\right)+4d_3\left(S_{v_3}+A_3\right)+3d_2\left(S_{v_2}+A_2\right)\right)}{r_1\left(\frac{d_1V_{c_1}}{P_{r_5}}+P_{c_1}\right)+3r_2\left(\frac{d_2V_{c_2}}{P_{r_4}}+P_{c_2}\right)+4r_3\left(\frac{d_3V_{c_3}}{P_{r_3}}+P_{c_3}\right)+3r_4\left(\frac{d_4V_{c_4}}{P_{r_2}}+P_{c_4}\right)}}$$

$$q_5 = \sqrt{\frac{2d_1\left(S_{v_1}+A_1\right)}{r_5\left(\frac{d_5V_{c_5}}{P_{r_1}}+P_{c_5}\right)}}$$

Now the Lagrangian function with multipliers λ_1, λ_2, λ_3 and λ_4

$L(q_1, q_2, q_3, q_4, q_5, \lambda) = P[\tilde{F}(\tilde{q})] - \lambda_1(q_2 - q_1) - \lambda_2(q_3 - q_2) - \lambda_3(q_4 - q_3) - \lambda_4(q_5 - q_4)$

We can obtain a solution from differentiating $L(q_1, q_2, q_3, q_4, q_5, \lambda_1, \lambda_2, \lambda_3, \lambda_4)$ with respect to q and equating to zero we get,

$$\frac{\partial L}{\partial q_1} = \frac{1}{12}\left[\frac{-d_5\left(S_{v_5}+A_3\right)}{q_1^2}+\frac{r_1}{2}\left(\frac{d_1V_{c_1}}{P_{r_5}}+P_{c_1}\right)\right]+\lambda_1$$

$$\frac{\partial L}{\partial q_2} = \frac{3}{12}\left[\frac{-d_4\left(S_{v_4}+A_4\right)}{q_2^2}+\frac{r_2}{2}\left(\frac{d_2V_{c_2}}{P_{r_4}}+P_{c_2}\right)\right]-\lambda_1+\lambda_2$$

$$\frac{\partial L}{\partial q_3} = \frac{1}{4}\left[\frac{-d_3\left(S_{v_3}+A_3\right)}{q_3^2}+\frac{r_3}{2}\left(\frac{d_3V_{c_3}}{P_{r_3}}+P_{c_3}\right)\right]-\lambda_2+\lambda_3$$

$$\frac{\partial L}{\partial q_4} = \frac{1}{4}\left[\frac{-d_2\left(S_{v_2}+A_2\right)}{q_4^2}+\frac{r_4}{2}\left(\frac{d_4V_{c_4}}{P_{r_2}}+P_{c_4}\right)\right]-\lambda_3+\lambda_4$$

$$\frac{\partial L}{\partial q_5} = \frac{1}{4}\left[\frac{-d_1\left(S_{v_1}+A_1\right)}{q_5^2}+\frac{r_5}{2}\left(\frac{d_5V_{c_5}}{P_{r_1}}+P_{c_5}\right)\right]+\lambda_4$$

$$\frac{\partial L}{\partial \lambda} = q_2 - q_1; \frac{\partial L}{\partial \lambda_2} = q_3 - q_2; \frac{\partial L}{\partial \lambda_3} = q_4 - q_3; \frac{\partial L}{\partial \lambda_4} = q_5 - q_4$$

$$q_1 = q_2 = q_3 = q_4 = q_5 = q^*$$

$$= \sqrt{\frac{2\left(d_5\left(S_{v_5}+A_5\right)+3d_4\left(S_{v_4}+A_4\right)+4d_3\left(S_{v_3}+A_3\right)+3d_2\left(S_{v_2}+A_2\right)+d_1\left(S_{v_1}+A_1\right)\right)}{r_1\left(\frac{d_1V_{c_1}}{P_{r_5}}+P_{c_1}\right)+3r_2\left(\frac{d_2V_{c_2}}{P_{r_4}}+P_{c_2}\right)+4r_3\left(\frac{d_3V_{c_3}}{P_{r_3}}+P_{c_3}\right)+3r_4\left(\frac{d_4V_{c_4}}{P_{r_2}}+P_{c_4}\right)+r_5\left(\frac{d_5V_{c_5}}{P_{r_1}}+P_{c_5}\right)}}$$

Hence we get q^*an optimal solution to problem.

5. NUMERICAL EXAMPLE

Consider any inventory system with following characteristics.

In which yearly demand is close to 1000 units, vendor's annual rate of production is nearer to 3200 units /year . The purchaser ordering cost per order is also close to 100$. The vendor's set up cost per set up is nearer to 400$ The unit production cost is nearer to 20. The annual inventory carrying cost per dollar invested in stocks is close to 0.2$. Determine the optimum order quantity.

Crisp Sense	Triangular Fuzzy Number	Pentagonal Fuzzy Number
Demand = 1000 units	$\tilde{d}=(d_1, d_2, d_3)$= (975,1000,1025)	$\tilde{d}=(d_1, d_2, d_3, d_4, d_5)$ =(950,975,1000,1025,1050)
setup cost = 400 $	$\tilde{S}_v = (S_{v_1}, S_{v_2}, S_{v_3})$ = (350,400,450)	$\tilde{S}_v = (S_{v_1}, S_{v_2}, S_{v_3}, S_{v_4}, S_{v_5})$ =(300,350,400,450,500)
Production cost = 20 $	$\tilde{V_c} = (V_{c_1}, V_{c_2}, V_{c_3})$ = (18,20 ,22)	$\tilde{V_c} = (V_{c_1}, V_{c_2}, V_{c_3}, V_{c_4}, V_{c_5})$ =(16,18,20,22,24)
Purchase cost = 25$	$\tilde{P_c} = (P_{c_1}, P_{c_2}, P_{c_3})$ = (20,25,30)	$\tilde{P_c} = (P_{c_1}, P_{c_2}, P_{c_3}, P_{c_4}, P_{c_5})$ =(15,20,25,30,35)
Production rate = 3200 units	$\tilde{P_r} = (P_{r_1}, P_{r_2}, P_{r_3})$ = (3100,3200,3300)	$\tilde{P_r} = (P_{r_1}, P_{r_2}, P_{r_3}, P_{r_4}, P_{r_5})$ =(3000,3100,3200,3300,3400)
Ordering cost = 100 $	$\tilde{A} = (A_1, A_2, A_2)$ = (85,100,115)	$\tilde{A} = (A_1, A_2, A_2, A_4, A_5)$ =(70,85,100,115,130)
Holding cost = 0.2$	$\tilde{r} = (r_1, r_2, r_3)$ = (0.1,0.2,0.3)	$\tilde{r} = (r_1, r_2, r_3, r_4, r_5)$ =(0,0.1,0.2,0.3,0.4)
Fuzzy order quantity	$\tilde{q}^* = (390.975, 390.975, 390.975)$	$\tilde{q}^* = (379.79, 379.79, 379.79, 379.79, 379.79)$
Minimum fuzzy joint relevant cost	$\tilde{F}(\tilde{q})$ = (1579.73,2500.65,3667.22)	$\tilde{F}(\tilde{q})$=(925.51,1597.52,2503.36,3648.31,5037.83)

6. CONCLUSION

This paper presents integrated inventory model under uncertain environment in two different cases using triangular and pentagonal fuzzy numbers. The defuzzification of the triangular and pentagonal fuzzy numbers was accomplished with the help of signed distance and graded mean methods respectively in order to derive the optimal solution. This serves to be an easy tool for the buyer and vendor in elevating the profit and depreciating the total cost.

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