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Lindley Approximate Bayes Estimation of Reliability Function in Weibull Model with Precautionary Loss

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Abstract
The two parameter Weibull distribution is a continuous distribution widely used in the study of reliability and life data. In this paper, we focus on different estimation approaches of two-parameter Weibull distribution based censored samples of lifetime data with type II censoring including, maximum likelihood (ML) and Bayesian estimation methodology of its Reliability Function. The ML estimation of the parameters and reliability function of Two parameter Weibull distribution is provided using the Newton–Raphson (NR) iterative method. The Bayesian estimates are provided via Lindley approximation. In the Bayesian estimation approach, for the shape and scale parameters, the Gamma prior is considered with Precautionary Loss Function. Finally, a simulated data set is analyzed for illustrative purposes to show the applicability of the proposed estimation methods. The performances of the ML and Bayesian estimates of reliability function are compared based mean squared errors (MSE) criteria.

Keywords: Weibull distribution, Reliability and life data, Bayesian estimation

1.Introduction

The weibull distribution introduced by the swedish physicist (Walooddi weibull, 1939). He used it to analyze the breaking strength of materials. Since then, it was widely used in reliability and life testing problems such as the time to failure or life length of a component, measured from some specified time until it fails.

In classical researches, the available data are considered as numbers. However, in real-world situations, some data are associated with an underlying imprecision due to inexactitude in the measuring process (human errors or machine errors), vagueness of the involved concepts or a certain degree of ignorance about the real values. The Bayesian estimation approach is of the main attractivein this situation. Sindhu et al. [33] discussed the Bayesian estimation of a mixture Gumbel models and their industrial application for process monitoring in a new format of control chart. The Bayesian inference of the mixture of two components of half-normal distribution based on both informative and noninformative priors are proposed by Sindhu et al. [34]. The posterior risks of the Bayesian estimators are compared to explore the effect of prior belief and loss functions. In lifetime analysis, the reliability function plays a principal role, which indicates how many parts are still in use after a certain running time and have not yet failed. The proposed approach is used in the reliability analysis using different types of IFFRs. One of the classic distributions to fit lifetime data is the Weibull distribution, which demonstrates some prominent properties. Several modifications of the Weibull distribution are considered by the authors. The Two parameter Weibull distribution with the shape and scale parameters has been extensively used in reliability and survival analysis.

Weibull distribution has been extensively used in life testing and reliability probability problems. The distribution is named after the Swedish scientist Weibull who proposed if for the first time in 1939 in connection with his studies on strength of material. Weibull (1951) showed that the distribution is also useful in describing the wear out of fatigue failures. Estimation and properties of the Weibull distribution is studied by many author's see Kao (1959).

The probability density function, reliability and hazard rate functions of Weibull distribution are given respectively as

$f(x) = p\theta x^{(p-1)} \exp(-\theta\, x^{P})$; $x, \theta, p > 0$ (1.1)

$R(t) = exp(-\theta t^{p})$; $t > 0$ (1.2)

$H(t) = p\theta t^{(p-1)}$; $t > 0$ (1.3)

Where $'\theta'$ is the scale and '*p*' is shape parameters.

The most widely used loss function in estimation problems is quadratic loss function given as $L(\hat{\theta}, \theta) = k(\hat{\theta} - \theta)^2$ where $\hat{\theta}$ is the estimate of θ, the loss function is called quadratic weighed loss function if k=1, we have

$L(\hat{\theta}, \theta) = (\hat{\theta} - \theta)^2$(1.4)

known as squared error loss function (SELF). This loss function is symmetrical because it associates the equal importance to the losses due to overestimation and under estimation with equal magnitudes however in some estimation problems such an assumption may be inappropriate. Overestimation may be more serious than underestimation or Vice-versa Ferguson (1985). Canfield (1970), Basu and Ebrahimi(1991). Zellner (1986) Soliman (2000) derived and discussed the properties of varian's (1975) asymmetric loss function for a number of distributions.

Norstrom (1996) introduced an alternative asymmetric precautionary loss function and also presented a general class of precautionary loss functions with quadratic loss function as a special case. These loss function approach infinitely near the origin to prevent underestimation and thus giving a conservative estimators, especially when, low failure rates are being estimated. These estimators are very useful and simple asymmetric precautionary loss function is

$\mathrm{L}(\hat{\theta}, \theta) = \frac{(\hat{\theta}-\theta)^2}{\hat{\theta}}$ (1.5)

where $\hat{\theta}$ is an estimate of θ .

The posterior expectation of the precautionary loss function in equation (1.5)is

$E_\pi\left[\frac{(\hat{\theta}-\theta)^2}{\hat{\theta}}\right] = E_\pi(\hat{\theta}) - 2E_\pi(\theta) + E_\pi\left(\frac{\theta^2}{\hat{\theta}}\right),$ (1.6)

The Bayes estimator $\hat{\theta}_{BP}$ of θ under precautionary loss function is the value of $\hat{\theta}$ which minimizes eqn.(1.6) is

$\hat{\theta}_{BPL} = [E_\pi(\theta^2)]^{\frac{1}{2}},$ (1.7)

Provided that $E_\pi(\theta^2)$ exists and is finite.

In a Bayesian setup, the unknown parameter is viewed as random variable. The uncertainty about the true value of parameter is expressedby

a prior distribution. The parametric inference is made using the posterior distribution which is obtained by incorporating the observed data in to the prior distribution using the Bayes theorem, the first theorem of inference. Hence we update the prior distribution in the light of observed data. Thus the uncertainty about the parameter prior to the experiment is represented by the prior distribution and the same after the experiment is represented by the posterior distribution. The various statistical models are considered.

The paper deals with the methods to obtain the approximate Bayes estimators of Reliability function of the Weibull distribution by using Lindley approximation technique for type-II censored samples. A bivariate prior density for the parametersand Precautionary loss function (PLF) are used to obtain the approximate Bayes Estimators. A statistical software R is used for numerical calculations for different approximate Bayes estimators and their relative mean squared errors by preparing programs to present the statistical properties of the estimators.

2.The Estimators

Let 'n' items that are put on test for their lives and the recorded lives are $y_1, y_2, \ldots \ldots \ldots y_n$ which follow a Weibull distribution with density given in equation (1.1) . The failure times are recorded as they occur until

a fixed number 'r' of times failed. Let $= (y_{(1)}, y_{(2)}, \ldots\ldots\ldots\ldots\ldots, y_{(n)})$, where $y_{(i)}$ is the life time of the i[th] item . Since remaining (n-r) items yet not failed thus have life times greater than $y_{(r)}$.
The likelihood function can be written as

$$L(y|\theta, p) = \frac{n!}{(n-r)!}(\mathrm{p}\theta)^r \prod_{i=1}^{r} y_i^{(p-1)} \exp(-\eta\theta), (2.1)$$

Where

$$\eta = \sum_{i=1}^{r} y_i^p + (n-r)y_r^p$$

The logarithm of the likelihood function is

$$\log L(y|\theta, p) \propto r \log p + r \log \theta + (p-1)\sum_{i=1}^{r} \log y_i - \eta\theta, (2.2)$$

assuming that 'p' is known, the maximum likelihood estimator $\hat{\theta}_{ML}$ of θ can be obtain by using equation (2.2) as

$$\hat{\theta}_{ML} = r/\eta (2.3)$$

If both the parameters p and θ are unknown their MLE's $\hat{p}_{ML}$ and $\hat{\theta}_{ML}$ can be obtained by solving the following equation

$$\frac{\delta}{\delta\theta}\log L = \frac{r}{\theta} - \eta = 0 \;, (2.4a)$$

$$\frac{\delta \log L}{\delta P} = \frac{r}{P} + \sum_{i=1}^{r} \log y_i - \theta\eta_1 = 0, (2.4b)$$

Where

$\eta_1 = \sum_{i=1}^{r} y_i^P \log y_i + (n-r)y_r^P \log y_r$, eliminating θ between the two equations of (2.4) and simplifying we get

$$\hat{p}_{ML} = \frac{r}{\eta^*} \; (2.5)$$

Where $\eta^* = \left[\frac{r\xi_1}{\eta} - \sum_{i=1}^{r} log y_i\right]$

Equation (2.5) may be solved for Newton-Raphson or any suitable iterative Method and this value is substituted in equation (2.4b) by replacing with p get $\hat{p}$ as

$$\hat{\theta}_{ML} = \frac{\frac{r}{\hat{p}_{ML}} + \sum_{i=1}^{r} log y_i}{\sum_{i=1}^{r} y_i^{\hat{p}_{ML}} \; log y_i + (\mathrm{n-r}) y_r^{\hat{p}_{ML}} \; log y_r}, (2.6)$$

The MLE's of R(t) and H(t) are given respectively by equation (1.2) and (1.3) after replacing θ and p by $\hat{\theta}_{ML}$ and $\hat{p}_{ML}$.

3. Bayes Estimator of θ when shape Parameter P is known

If p is known assume gamma prior $\gamma(\alpha, \beta)$ as conjugate prior for θ as

$$g\left(\theta\middle|\underline{y}\right) = \frac{\beta^\alpha}{\Gamma\alpha}(\theta)^{(\alpha+1)} \exp(-\beta\theta)\,; (\alpha, \beta) > 0, \theta > 0 \;, (3.1)$$

The posterior distribution of θ using equation (2.1) and (3.1) we get

$$h\left(\theta\middle|\underline{y}\right) = \frac{(\eta+\beta)^{r+\alpha}}{\Gamma(r+\alpha)}(\theta)^{(r+\alpha-1)} \exp\left(-\theta(\eta+\beta)\right), (3.2)$$

Under General Precautionary Loss Function, the Bayes estimator $\hat{\theta}_{BPL}$ of θ using (1.9) and (3.2) given by

$$\hat{\theta}_{BPL} = \left[\frac{(r+\alpha)(r+\alpha+1)}{(\eta+\beta)}\right]^{\frac{1}{2}} (3.4)$$

Bayes Estimator of R(t)

The posterior distribution of R using equation (1.2) and (3.2), is given as

$$h(R|t) = \frac{[\,c\,(\eta+\beta)]^{(r+\alpha)}}{\Gamma(r+\alpha)}(-\log R)^{(r+\alpha-1)} R^{(\,c\,(\eta+\beta)\,-\,1)} dR; (3.6)$$

Where $c = t^{-p}$

The Bayes estimator of R(t) under precautionary loss function

$$\hat{R}_{BPL} = \left[1 + \frac{2}{(\eta+\beta)}\right]^{(r+\alpha)} ; (3.7)$$

3.The Bayes estimators with θ and p unknown:

The joint prior density of θ and p is given by

$$G(\theta|p) = g_1(\theta|p).\, g_2(p)$$

$$G(\theta|p) = \frac{1}{\lambda\Gamma\xi} p^{-\xi} \theta^{(\xi-1)}.\exp\left[-\left(\frac{\theta}{p}+\frac{p}{\lambda}\right)\right] \ ; (\theta, p, \lambda, \xi) > 0, (4.1)$$

where

$$g_1(\theta|p) = \frac{1}{\Gamma\xi} p^{-\xi}\theta^{(\xi-1)}.\exp\left[-\frac{\theta}{p}\right];(4.2)$$

And

$$g_2(p) = \frac{1}{\lambda}\exp\left(-\frac{p}{\lambda}\right) \quad ; \ (4.3)$$

The joint posterior density of θ and p is

$$h^*\left(\theta, p\middle|\underline{y}\right) = \frac{\frac{1}{\lambda\Gamma\xi}p^{-\lambda}\theta^{(\xi+1)}\exp\left[-\left\{\frac{\theta}{p}+\frac{p}{\lambda}\right\}\right](p\theta)^r \prod_{i=1}^{r} x_i^{(p-1)} e^{-p\theta}}{\iint \frac{1}{\lambda\Gamma\xi}p^{(r-\xi)}\theta^{(r+\xi+1)}\prod_{i=1}^{r} x_i^{(p-1)}.\exp\left[-\left\{\frac{\theta}{p}+\frac{p}{\lambda}+p\theta\right\}\right]d\theta dp}; \quad (4.4)$$

Approximate Bayes Estimators

The Bayes estimators of a function $\mu = \mu(\theta, p)$ of the unknown parameter θ and p under squared error loss is the posterior mean

$$\hat{\mu}_{ABS} = E\left(\mu\middle|\underline{x}\right) = \frac{\iint \mu(\theta,p)G(\theta,p|\underline{x})d\theta dp}{\iint G(\theta,p|\underline{x}).d\theta.dp} \ ;(4.5)$$

To evaluate (4.5) consider the method of Lindley approximation

$$E\left(\mu(\theta,p)\middle|\underline{x}\right) = \frac{\int \mu(\theta).e^{(l(\theta)+\rho(\theta))}d\theta}{\int e^{(l(\theta)+\rho(\theta))}d\theta} \ ;(4.6)$$

Where $(\theta) = \log g(\theta)$, and $g(\theta)$ is an arbitrary function of θ and $l(\theta)$ is the logarithm likelihood function

The Lindley approximation for two parameter is given by

$$E\left(\hat{\mu}(\theta,p)\middle|\underline{x}\right) = \mu(\theta,p) + \frac{A}{2} + \rho_1 A_{12} + \rho_2 A_{21} + \frac{1}{2}[l_{30}B_{12} + l_{21}C_{12} + l_{12}C_{21} + l_{03}B_{21}],(4.7)$$

Where

$A = \sum_1^2\sum_1^2 \mu_{ij}\sigma_{ij}$; $l_{\eta\epsilon} = \left(\delta^{(\eta+\epsilon)} l\middle|\delta\theta_1^{\eta}\delta\theta_2^{\epsilon}\right)$; where$(\eta+\epsilon) = 3$ for $i,j = 12$, $\rho_i = (\delta\rho|\delta\theta_i)$;

$$\mu_i = \frac{\delta\mu}{\delta\theta_i} \ ; \quad \mu_{ij} = \frac{\delta^2\mu}{\delta\theta_i\delta\theta_j} \ ; \forall i \neq j \ ;$$

$$A_{ij} = \mu_i\sigma_{ij} + \mu_j\sigma_{ji} \ ; \ B_{ij} = \left(\mu_i\sigma_{ii} + \mu_j\sigma_{ij}\right)\sigma_{ii} \ ;$$

$$C_{ij} = 3\mu_i\sigma_{ii}\sigma_{ij} + \mu_j\left(\sigma_{ii}\sigma_{jj} + 2\sigma_{ij}^2\right);$$

Where σ_{ij} is the (i,j)th element of the inverse of matrix $\{-l_{jj}\}; i,j = 1,2$ s.t. $l_{ij} = \frac{\delta^2 l}{\delta\theta_i\delta\theta_j}$.

All the above functions are evaluated at MLE of (θ_1, θ_2).In our case $(\theta_1\theta_2) = (\theta, p); So\ \mu(\theta) = \mu(\theta, p)$

To apply Lindley approximation (4.5), we first obtain σ_{ij} , elements of the inverse of $\{-l_{jj}\}; i,j = 1,2$,which can be shown to be

$$\sigma_{11} = \frac{M}{D}, \ \sigma_{12} = \sigma_{21} = \frac{\delta_1}{D}, \sigma_{22} = \frac{r}{D\,\theta^2},(4.8a)$$

Where$M = \left(\frac{r}{p^2} + \theta\delta_2\right)$;$D = \left[\frac{r}{\theta^2}\left(\frac{r}{p^2} + \theta^2\delta_2\right)\right]$; (4.8b)

$$\delta_2 = \sum_{i=1}^{r} x_i^p (log x_i)^2 + (n-r)x_r^p(log x_r)^2; \qquad (4.8c)$$

To evaluateρ_i , take the joint prior $G(\theta|p)$

$$G(\theta|p) = \frac{1}{\lambda\Gamma\xi}p^{-\xi}\theta^{(\xi-1)}.\exp\left[\left\{-\frac{\theta}{p}+\frac{p}{\lambda}\right\}\right] \ ; (\theta, p, \lambda, \xi) > 0, (4.9)$$

$$\Rightarrow \rho = \log[G(\theta|p)] = constant - \xi log p - (\xi-1)log\theta - \frac{\theta}{p} - \frac{p}{\lambda}$$

Therefore

$$\rho_1 = \frac{\partial\rho}{\partial\theta} = \frac{(\xi-1)\theta}{\theta} - \frac{1}{p};(4.9a)$$

and

$$\rho_2 = \frac{\theta}{p^2} - \frac{1}{\lambda} - \frac{\xi}{p};(4.9b)$$

Further more

$l_{21} = 0\,; l_{12} = -\delta_2\,; l_{03} = \frac{2r}{p^3} - \theta\delta_3;$ (4.9c)

$and\ \ l_{30} = \frac{2r}{\theta^3};$ (4.9d)

Where $\delta_3 = \sum_{i=1}^{r} x_i^p\,(logx_i)^3 + (n-r)x_r^p(logx_r)^3$

By substituting above values in eqn. (4.7), yields the Bayes estimator under SELF using Lindley approximation denoted by $\hat{\mu}_{ABS}$

$\hat{\mu}_{ABS} = E(\mu(\theta,p)) = \mu(\theta,p) + Q + \mu_1 Q_1 + \mu_2 Q_2;$ (4.10)

Where Q $=\frac{1}{2}[\mu_{11}\sigma_{11} + \mu_{21}\sigma_{21} + \mu_{12}\sigma_{12} + \mu_{22}\sigma_{22}]$;(4.10a)

$Q_1 = \frac{1}{\theta^2 D^2}[\frac{M\theta D}{p}(p(\xi-1)-1) + \frac{\theta^2\delta_1 D}{\lambda p^2}\{\lambda\theta - p^2 - \lambda\xi p\}$

$+\frac{rM^2}{\theta} - \frac{rM\delta_1}{2} - \theta^2\delta_1^2\delta_2 + \frac{r^2}{p^3}\delta_1 - \frac{\theta r\delta_1\ \delta_3}{2}];$ (4.10b)

$Q_2 = \frac{1}{\theta^2 D^2}[\frac{\theta\ \ \delta_1 D}{p}(p(\xi-1)-\theta) + \frac{rD}{\lambda p^2}\{\lambda\theta - p^2 - \lambda\xi p\}$

$+\ \frac{rM\delta_1}{\theta} - \frac{3\delta_1 r\delta_2}{2} + \frac{r^2}{\theta^2 p^3} - \frac{r^2\delta_3}{2\theta}];$ (4.10c)

All the function of right hand side of the equation(4.10) are to be evaluated for $\hat{\theta}_{ML}$ and $\hat{p}_{ML}$.

5. Approximate Bayes Estimators under Precautionary loss function

The Approximate Bayes estimator of a function $\mu = \mu(\theta,p)$ of unknown parameters $\theta\ \ and\ \ p$ under PLF in equation(1.7) is given by

$\hat{\mu}_{ABP} = [E_h(\mu^2)]^{\frac{1}{2}}$(5.1)

Where

$E_{h^*}(\mu^2|\bar{x}) = \frac{\iint \mu^2\ h^*(\theta,p)d\theta\ dp}{\iint\ h^*(\theta,p)d\theta\ dp}$;(5.2)

Special Cases

(i) Let $\mu(\theta,p) = \frac{1}{R}$;

The approximate Bayes estimator of R under Precautionary loss functionis

$\hat{R}_{ABPL} = R\left[1 + \frac{t^p}{\theta\ \ D}\phi_5 - 2t^p(Q_1 + \theta\ logt\ Q_2)\right]^{\frac{1}{2}}$; at $(\hat{\theta}_{ML}, \hat{p}_{ML})$,(5.3)

Where

$\phi_5 = [2\theta t^p M + (2\theta t^p - 1)logt(2\theta\delta_1 - rlogt)^2]$

6. Numerical Calculations and Comparison

The numerical calculations are done by using R Language programming and results are presented in form of tables.

1: The values of and are generated from the equations (4.2 - 4.3) for given λ=2, and ξ=3, which comes out to be θ=0.238 and p=0.227. For these values of and p the Weibull random variates are generated.

2: Taking the different sizes of samples n=25 (25) 100 with failure censoring, MLE's, the Approximate Bayes estimators ,and their respective MSE's (in parenthesis) by repeating the steps 500 times, are presented in the tables from (1), for t=2, R(t)=0.7568 and parameters of prior distribution α=2 ,and β =3.

5: Table(1) presents the MLE of R(t) and Approximate Bayes estimators of reliability function R(t) of Weibull density under PLF (for and p both unknown) with their respective MSE's. The all four estimators are efficient for larger sample size but as sample approaches to 100 their MSE's started increasing.

Table(1) Mean and MSE's ofR(t)

$(\lambda = 2, \xi = 3, = 0.238, p = 0.227, t = 2, R(t) = .07568)$

n	r	$\hat{R}_{ML}$	$\hat{R}_{ABS}$	$\hat{R}_{ABP}$
25	20	0. 649642	0. 74576	0. 843621

		(1.7239x10^{-5})	**(4.0544x10^{-5})**	**(2.5522x10^{-6})**
50	30	0.713952	0. 750967	0. 852534
		(1.63169x10^{-7})	**(3.67188x10^{-6})**	**(4.2797 x10^{-7})**
75	50	0. 740252	0. 755524	0. 875134
		(2.7167x10^{-8})	**(4.0074x10^{-8})**	**(4.9881x10^{-8})**
100	75	0. 768581	0. 8482655	0. 923851
		(7.1028x10^{-6})	**(7.0887 x10^{-5})**	**(7.6543 x10^{-4})**

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